Some New Approaches to Some Separation Axioms in Bispaces

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Abstract: In this paper we have studied some new notions in Bispaces like strongly pairwise hausdroff, strongly pairwise regular, strongly pairwise normal. Also studied how the pairwise intersection property are useful for these above properties.

Keyword: Bispace, pairwise intersection property, pairwise hausdroff, pairwise regular, pairwise normal .

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INTRODUCTION:

The notion of a topological space was generalized to a σ-space(or simply space or Alexandroof space) by A.D.Alexandroof [1] weakening the union requirements. J.C.Kelly [4] introduced the idea of a bitopological space. The concept of σ-space was used by Lahiri and Das [5] to generalize the notion of a bitopological space to a bispace where several ideas like pairwise Hausdorffness, pairwise bicompactness etc. were also studied. Some new bitopological notions like strong separation axioms, strong pairwise Housdroffnesss etc. was studied by A. Mukherjee [6]. The triple (X, τ_1, τ_2) where (X, τ_1) and (X, τ_2) are two σ-space on X, is called a bispace. Due to presence of two σ- spaces in a bispace, it is always possible to consider the interior of a (τ_i)open set with respect τ_j ,where$i, j \in \{1,2\}, i \neq j$. Now it is interesting to note, for a nontrivial (τ_i)open set U, $(\tau_j)intU$ may even be an empty set. Even $(\tau_j)intU \neq \emptyset$, it is obvious that $(\tau_j)intU \subset U$. This observation leads us to define the notion of strong pairwise compactness. In subsequent endeavors, we introduce the strong separation axioms, strong pairwise Hausdroffness, strong pairwise regularity, strong pairwise normality. In fact, the results of bispace are generalization of the results of σ-space. But the notions we introduce here are not generalization of any result of σ-space since nontrivial similar concepts in σ-space are absurd. However when $\tau_1 = \tau_2$ in a bispace, these notions reduce to equivalent conventional concept of σ-space.

Throughout the paper, a bispace(X, τ_1, τ_2) is simply denoted by X. Rdenotes the set of real numbers and N, the set of natural numbers and Q, the set of rational numbers.

2. PRELIMINARIES:

Definition 2.1[1]: A set X is called anσ-space or Alexandroff space or simply a space if in it is chosen a system F of subsets satisfying the following axioms:

1) The intersection of a countable number of sets from F is a set in F.
2) The union of a finite number of sets from F is a set in F.

3) The void set φ is a set in F.
4) The whole set X is a set in F.

Sets of F are called closed sets. Their complementary sets are called open. It is clear that instead of closed sets in the definition of the space one may put open sets with subject to the conditions of countable summability, finite intersectibility and the condition that X and void set φ should be open. The collection of all such open sets will sometimes be denoted by τ and the space by (X,τ). Note that a topological space is a space but in general τ is not a topology as can be easily seen by taking $X = R$ and τ as the collection of all F_σ sets in R.

Definition 2.2[1]: To every set M of a space (X,τ) we correlate its closure $\bar{M}$, the intersection of all closed sets containing M. The closure of a set M will be denoted by $(\tau)cl(M)$ or simply clM when there is no confusion about τ.

Generally, the closure of a set in a space may not be a closed set. The definition of limit point of a set is parallel as in the case of a topological space.

From the axioms, it easily follows that

1) $\overline{M \cup N} = \bar{M} \cup \bar{N};$
2) $M \subset \bar{M};$
3) $\bar{M} = \bar{\bar{M}};$
4) $\bar{\varphi} = \varphi.$
5) $\bar{A} = A \cup A'$

whereA' denotes the set of all limit point of A.

Definition 2.3[5]: The interior of a set M in a space (X, τ) is defined as the union of all open sets contained in M and is denoted by $(\tau)intM$or $intM$ when there is no confusion about τ.

Definition 2.4[5]: Let X be a nonempty set. If τ_1 and τ_2 be two collections of subsets of X such that (X, τ_1) and (X, τ_2) are two σ-spaces, then X is called a bispace and denoted by (X, τ_1, τ_2).

Definition 2.5[5]: A bispace X is said to be pairwise Hausdorff if for each pair of distinct points x and y of X, there exist $U \in \tau_i$ and $V \in \tau_j$ such that $x \in U, y \in V$ and $U \cap V = \emptyset$.

Definition 2.6[5]: In a bispace X, τ_i is said to be regular with respect to τ_j if for each $x \in X$ and each (τ_i)closed A with $x \notin A$, there exist $U \in \tau_i$and $V \in \tau_j$ such that $x \in U, A \subset V$ and $U \cap V = \emptyset$. $X$is said to be pairwise regular if τ_i is said to be regular with respect to τ_j for $i, j = 1,2, i \neq j$.

Definition 2.7[5]: A bispace X, is said to be pairwise normal if for any pair of a (τ_i)closed A and a (τ_j)closed B with $A \cap B = \emptyset$, there exist $U \in \tau_i$and $V \in \tau_j$ such that $A \subset U, B \subset V$ and $U \cap V = \emptyset$.

3. Strong separation axioms in Bispaces:

In this section, we introduce the notions of strong separation axioms in bispaces.

Definition 3.1: A bispace X is said to satisfy the pairwise intersection property if for each pair of a (τ_i)open set $A(\neq X)$ and a (τ_j)open set $B(\neq X)$ with $A \cap B \neq \emptyset$, we have a (τ_j)open set U and a (τ_i)open set V such that $A \cap B \subset U \subset A$ and $A \cap B \subset V \subset B$.

Example 3.1: Let $X = [0,3]$. Let $\{G_i\}$ be collection of all countable subsets in $[0,1] - Q$ and $\{F_i\}$ be collection of all countable subsets in $[2,3] - Q$.

Let $\tau_1 = \{X, \emptyset, \{\sqrt{2}\}, G_i \cup \{\sqrt{2}\}\}$and $\tau_2 = \{X, \emptyset, \{\sqrt{2}\}, F_i \cup \{\sqrt{2}\}\}$. Then (X, τ_1, τ_2) is a bispace. Here τ_1 and τ_2 are independent, and the bispace (X, τ_1, τ_2) satisfies the pairwise intersection property.

Definition 3.2: A bispace X is said to be strongly pairwise Hausdorff if for each pair of distinct points x and $y$of X, there exist $U \in \tau_i$ and $V \in \tau_j$ such that $x \in (\tau_j)intU, y \in (\tau_i)intV$ and $U \cap V = \emptyset$.

It readily follows that a strongly pairwise Hausdroff space X is pairwise Hausdroff but the converse is not true.

Example 3.2: Let $X = [0,2] - Q$. Let $\{G_i\}$ be collection of all countable subsets in $[0,1] - Q$ and $\{F_i\}$ be collection of all countable subsets in $[1,2] - Q$.

Let $\tau_1 = \{X, \emptyset, G_i\}$and $\tau_2 = \{X, \emptyset, F_i\}$. Then (X, τ_1, τ_2) is a bispace. Here τ_1 and τ_2 are independent, and the bispace (X, τ_1, τ_2) is pairwise Hausdroff but not strongly pairwise Hausdroff.

Theorem 3.1: A pairwise Hausdroffbispace with pairwise intersection property is strongly pairwise Hausdroff.

Proof: Let X be a pairwise Hausdroff bispace with pairwise intersection property. Suppose $x, y \in X, x \neq y$. Now by pairwise Hausdorffness, there exist a (τ_1)open set U_1 and a (τ_2)open set V_1 such that $x \in U_1, y \in V_1$with $U_1 \cap V_1 = \emptyset$. Also there exist a (τ_2)open set V_2 and a (τ_1)open set U_2 such that $x \in V_2, y \in U_2$with $U_2 \cap V_2 = \emptyset$.

Now $U_i \cap V_j \neq \emptyset$. So we obtain a (τ_j)open set G_j and a (τ_i)open set H_i such that $U_i \cap V_j \subset G_j \subset U_i$and $U_i \cap V_j \subset H_i \subset V_j$. Thus we get, $x \in (\tau_2)intU_1$ and $y \in (\tau_1)intV_1$with $U_1 \cap V_1 = \emptyset$.

Also we obtain, $x \in (\tau_1)intV_2$ and $y \in (\tau_2)intU_2$with $U_2 \cap V_2 = \emptyset$. Thus the bispace is strongly pairwise Hausdorff.

Definition 3.3: A bispace X is said to be strongly pairwise regular if for each $x \in X$ and each (τ_i)closed set F with $x \notin F$, there exist a (τ_i)open set U and (τ_j)open set V such that $x \in U$ and $F \subset (\tau_i)intV$ with $U \cap V = \emptyset$.

It is obvious that if a bispace X is strongly pairwise regular, then X is pairwise regular.

Theorem 3.2: A bispace X with pairwise intersection property is strongly pairwise regular if X is pairwise regular, and for each $i \in \{1,2\}, (X, \tau_i)$ is regular.

Proof: Let $x \in X$ and each (τ_i)closed set $F$with $x \notin F$. Hence by pairwise regularity, there exist a (τ_i)open set U_1 and a (τ_2)open set V_1 such that $x \in U_1, F \subset V_1$with $U_1 \cap V_1 = \emptyset$.

Also (X, τ_i) is regular. Hence there exist (τ_i)open sets $U_1{}'$ and $V_1{}'$ such that $x \in U_1{}', F \subset V_1{}'$ with $U_1{}' \cap V_1{}' = \emptyset$. Thus $x \in U_1 \cap U_1{}' \in \tau_i$and $F \subset V_1 \cap V_1{}'$.

Now by pairwise intersection property, there exists a(τ_i)open set H such that $V_1 \cap V_1{}' \subset H \subset V_1$. We put $U = U_1 \cap U_1{}'$and $V = V_1$. Then $x \in U$ and $F \subset (\tau_i)intV$. It is easy to see that $U \cap V = \emptyset$.

Hence X is strongly pairwise regular.

Definition 3.4: A bispace X is said to be strongly pairwise normal if for each (τ_i)closed E and each (τ_j)closed F with $E \cap F = \emptyset$, there exist a (τ_j)open set U and a (τ_i)open set V such that $E \subset (\tau_i)intU, F \subset (\tau_j)intV$ and $U \cap V = \emptyset$.

It follows that a strongly pairwise normal bispace$X$ is pairwise normal but for each $i \in \{1,2\}, (X, \tau_i)$ need not be normal. This result holds in bitopological space([6] , example 2.5 and 2.6.). So it is also true for bispace.

Theorem 3.3: If a bispace (X, τ_1, τ_2) is strongly pairwise normal, and for each $i \in \{1,2\}, (X, \tau_i)$ satisfies the axiom T_1, then $\tau_1 = \tau_2$.

Proof: Straightforwed.

We recall that a σ-space (X, τ) is normal if for nonempty (τ)closed sets $E, F$with $E \cap F = \emptyset$, there exist (τ)open sets U, V such that $E \subset U, F \subset V$ and $U \cap V = \emptyset$.

In this fashion, we say that a σ-space (X, τ) possesses 'covering properties of closed sets by open sets' if for each nontrivial (τ)closed set E there exists a (τ)open set $U(\neq X)$ such that $E \subset U$.

Theorem 3.4: A bispace X with pairwise intersection property is strongly pairwise normal if X is pairwise normal and for each $i \in \{1,2\}, (X, \tau_i)$ possesses covering properties of closed sets by open sets.

Proof: Let E be (τ_i)closed and F be (τ_j)closed set with $E \cap F = \emptyset$. Hence by pairwise normality, there exist a (τ_j)open set U and (τ_i)open set V such that $E \subset U, F \subset V$and $U \cap V = \emptyset$.

Since each of (X, τ_i) and (X, τ_j) possesses covering properties of closed sets by open sets, we obtain a (τ_i)open set G_i and a (τ_j)open set H_j such that $E \subset G_i, F \subset H_j$.

Thus there exist a (τ_i)open set G and a (τ_j)open set H such that $U \cap G_i \subset G \subset U$and $V \cap H_j \subset H \subset V$. So we have $E \subset (\tau_i)intU$ and $F \subset (\tau_j)intV$with $U \cap V = \emptyset$. Thus the bispace is strongly pairwise normal.

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