Stability of the Centre of Mass of Two Satellites Connected by An Extensible String Under Air Resistance, Solar Pressure and Oblateness of the Earth in Circular Orbit

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Abstract : This paper is to study the stability of equilibrium position of the centre of mass of two satellites connected by an extensible string under the influence of air resistance and the shadow of the oblate earth due to solar pressure in the central gravitational field of the earth in circular orbit. We have obtained an equilibrium position and has been shown to be stable in the sense of Liapunov.

Key words : Satellites, Air resistance, oblateness of the earth and solar pressure.

I. Introduction

The present paper is devoted to examine the stability of an equilibrium point of the centre of mass of the system of two satellites connected by an extensible string which is supposed to be light and flexible under the influence of perturbative forces mentioned in the Abstract in case of circular orbit. By exploiting Lagrange's equations of motion of first kind, we have derived differential equations of motion of one of the two satellites when their centre of mass is moving along a keplrian orbit in Nechvill's coordinate system. The general solution of the differential equations obtained are beyond our reach. Hence in order to facilitate our problem, we put e=0 and so $\rho = 1$ and $\rho^1 = 0$. Then we get the equations of motion in circular orbit for the centre of mass of the system. Then Jacobi's integral for the problem in case of two dimensional motion has been obtained. After that we get an equilibrium position of the centre of mass of the system and has been found to be stable in the sense of Liapunov him.

II. Equations of motion of the system in Elliptic orbit in Nechvill's Coordinate System.

The equations of motion of one of the two satellites with respect to the centre of mass in Nechvill's coordinate system in elliptic orbit have been obtained in the form:

$$x''-2y'-3\rho x+\frac{4Bx}{\rho}+A\psi_1\rho^3\cos\in\cos(v-\alpha)+f\rho\rho' = -\overline{\lambda_\alpha}\rho^4\left[1-\frac{\ell_0}{\rho\sqrt{x^2+y^2+z^2}}\right]x$$

$$y''-2x'-\frac{By}{\rho}-A\psi_1\rho^3\cos\in\sin(v-\alpha)+f\rho^2 = -\overline{\lambda_\alpha}\rho^4\left[1-\frac{\ell_0}{\rho\sqrt{x^2+y^2+z^2}}\right]y \qquad \text{-------- (1)}$$

$$\text{and} \qquad z''+z-\frac{Bz}{\rho}-A\psi_1\rho^3\sin\in = -\overline{\lambda_\alpha}\rho^4\left[1-\frac{\ell_0}{\rho\sqrt{x^2+y^2+z^2}}\right]z$$

Where

$$f = \frac{a_1 p^3}{\sqrt{\mu p}} = \text{Air resistance parameter}$$

$$A = \frac{p^3}{\mu}\left[\frac{B_1}{m_1} - \frac{B_2}{m_2}\right] = \text{Solar pressure parameter}$$

$$B = \frac{3K_2}{p_2} = \text{oblateness parameter}$$

$$\psi_1 = \text{shadow function parameter}$$

$$\overline{\lambda_\alpha} = \frac{p^3}{\mu}\lambda_\alpha;$$

$$\rho = \frac{1}{1+e\cos v}$$; V being true anamaly of the orbit of the centre of mass of the system.

ℓ_0 = Natural length of the string connecting the two satellites of masses m_1 and m_2

Here dashes denotes differentiations with respect to true anomaly v

The condition of constraint is given by

$$x^2 + y^2 + z^2 \le \frac{\ell_0^2}{\rho^2} \qquad \text{-------- (2)}$$

Putting ρ=1 and ρ^1=0 for e=0 in (1), we get two dimensional equations of motion in the form.

$$\left.\begin{aligned} & x'' - 2y' - 3x + 4Bx + A\psi_1 \cos\in \cos(v-\alpha) = -\overline{\lambda_\alpha}\left[1-\frac{\ell_0}{r}\right]x \\ & and \quad y'' + 2x' - By + f - A\psi_1 \cos\in \sin(v-\alpha) = -\overline{\lambda_\alpha}\left[1-\frac{\ell_0}{r}\right]y \end{aligned}\right\} \qquad \text{---------- (3)}$$

Where $r = \sqrt{x^2+y^2}$

The condition of constant given by (2) takes the form:

$$x^2 + y^2 \le \ell_0^2$$

The periodic terms in (3) can be averaged by the following relations:

$$\left.\begin{aligned} & \frac{1}{2\pi}\left[\int_{-\theta\atop \psi_1=0}^{\theta} A\cos\in\cos(v-\alpha)dv + \int_{\theta\atop \psi_1=0}^{2\pi-\theta} A\cos\in\cos(v-\alpha)dv\right] = \frac{-A\cos\in\cos\alpha\sin\theta}{\pi} \\ & and \\ & \frac{1}{2\pi}\left[\int_{-\theta\atop \psi_1=0}^{\theta} A\cos\in\sin(v-\alpha)dv + \int_{\theta\atop \psi_1=0}^{2\pi-\theta} A\cos\in\sin(v-\alpha)dv\right] = \frac{-A\cos\in\sin\alpha\sin\theta}{\pi} \end{aligned}\right\} \text{----- (4)}$$

Where θ is taken to be constant

Thus, using (4) the system of equations of motion (3) becomes

$$\left.\begin{aligned} & x'' - 2y' - 3x + 4Bx - \frac{A\cos\in\cos\alpha\sin\theta}{\pi} = -\overline{\lambda_\alpha}\left[1-\frac{\ell_0}{r}\right]x \\ & and \quad y'' + 2x' - By + f - \frac{A\cos\in\cos\alpha\sin\theta}{\pi} = -\overline{\lambda_\alpha}\left[1-\frac{\ell_0}{r}\right]y \end{aligned}\right\} \qquad \text{----------(5)}$$

We find that the equations of motion given by (5) do not contain time t explicitly. Hence, there must exist Jacobi's integral for the problem.

Multiplying the first equation of (5) by $2x^1$ and the second equation of (5) by $2y^1$ and adding them together and then integrating, we get Jacobi's integral in the form.

$$\begin{aligned}&x'^2+y'^2-3x^2+4Bx^2-By^2-\frac{2Ax\cos\in\cos\alpha\sin\theta}{\pi}-\frac{2Ay\cos\in\sin\alpha\sin\theta}{\pi}\\&+2fy+\overline{\lambda_\alpha}\left[\left(x^2+y^2\right)-2\ell_0\sqrt{x^2+y^2}\right]=h\end{aligned} \quad \text{----------- (6)}$$

Where h is the constant of integration

III. Equilibrium position of the system:

We have derived the system of equations (5) for the motion of the system in rotating frame of reference. It has been assumed that the system is moving with effective constraints and the connecting cable of the two satellites will always remain tight.

The equilibrium positions of the system are given by the constant values of the co-ordinates in the rotating frame of reference.

Let $x = x_0$ and $y=y_0$ give the equilibrium position where x_0 and y_0 are constants

Hence, x'=0=x" and y'=0=y"

Thus, equations given by (5) take the form:

$$\left.\begin{aligned}&-3x_0+4Bx_0-\frac{A}{\pi}\cos\in\cos\alpha\sin\theta=-\overline{\lambda_\alpha}\left[1-\frac{\ell_0}{r_0}\right]x_0\\&-By_0+f-\frac{A}{\pi}\cos\in\sin\alpha\sin\theta=-\overline{\lambda_\alpha}\left[1-\frac{\ell_0}{r_0}\right]y_0\\&where\quad r_0=\sqrt{x_0^2+y_0^2}\end{aligned}\right\} \quad \text{----------- (7)}$$

We find that it is very difficult to get the solution of equations of motion (7) in its present form. Hence, we put $\alpha = 90^0$ and so the sun rays are assumed to be in the direction perpendicular to the line of perigee of the circular orbit of the centre of mass of the system.

Hence on putting $\alpha = 90^0$ in (7), we get

$$\left.\begin{aligned}&-(3-4B)x_0=-\overline{\lambda_\alpha}\left[1-\frac{\ell_0}{r_0}\right]x_0\\&-By_0+f-\frac{A\cos\in\sin\theta}{\pi}=-\overline{\lambda_\alpha}\left[1-\frac{\ell_0}{r_0}\right]y_0\end{aligned}\right\} \quad \text{------------- (8)}$$

From 8, we get the equilibrium point as

$$\left[0,\frac{\frac{A}{\pi}\cos\in\sin\theta+\overline{\lambda_\alpha}\ell_0-f}{\overline{\lambda_\alpha}-B}\right] \quad \text{------------ (9)}$$

It can be easily seen that the equilibrium position (9) gives a meaningful value of Hook's modulus of elasticity if $\left(\frac{A\cos\in\sin\theta}{\pi}-f\right)$ is positive.

IV. Stability of the equilibrium point of the system

We examine the stability of the equilibrium point given by (9) of the system in the sense of Liapunov.

For this,

Let a = x = 0 and $b = y = \dfrac{\dfrac{A\cos\in\sin\theta}{\pi} + \overline{\lambda_\alpha}\ell_0 - f}{\overline{\lambda_\alpha} - B}$

Let η_1 and η_2 be the small variations in x_0 and y_0 respectively. For the given position of equilibrium (o, b) given by (9). We have

$$\left.\begin{aligned} &x = 0 + \eta_1 and \;\; y = b + \eta_2 \\ &\therefore x' = \eta_1' and \;\; y' = \eta_2' \\ &x'' = \eta_1'' and \;\; y'' = \eta_2'' \end{aligned}\right\} \qquad \text{----------} \; (10)$$

Putting $\alpha=90^0$ in equations (5) and using (10), we get

$$\left.\begin{aligned} &\eta_1'' - 2\eta_2' - (3-4B)\eta_1 = -\overline{\lambda_\alpha}\left[1 - \frac{\ell_0}{r}\right]\eta_1 \\ &and\; \eta_2'' + 2\eta_1' + f - \frac{A\cos\in\sin\theta}{\pi} = -\overline{\lambda_\alpha}\left[1 - \frac{\ell_0}{r}\right](b+\eta_2) \end{aligned}\right\} \qquad \text{-----------} \; (11)$$

Where $r = \sqrt{\eta_1^2 + (b+\eta_2)^2}$

Multiplying the first equation of (11) by $2\eta_1'$ and the 2^{nd} equation of (11) by $2(b+\eta_2)'$ respectively and adding these together and then integrating, we get Jacobi's integral for the problem in the form.

$$\begin{aligned} &\eta_1'^2 + \eta_2'^2 - (3-4B)\eta_1^2 - B(b+\eta_2)^2 + 2(b+\eta_2) f \\ &\qquad - \frac{2A(b+\eta_2)\cos\in\sin\theta}{\pi} + \overline{\lambda_\alpha}\left[\eta_1^2 + (b+\eta_2)^2\right] \\ &\qquad - 2\overline{\lambda_\alpha}\ell_0\left[\eta_1^2 + (b+\eta_2)^2\right]^{1/2} = h \end{aligned} \qquad \text{----------} \; (12)$$

Where h is the constant of integration.

To examine the stability of the equilibrium point in the sense of Liapunov, we take Jacobi integral given by (12) as Liapunov's function v (η_1', η_2', η_1, η_2) and is obtained expanding the terms of (12) as

$$\begin{aligned} v(\eta_1', \eta_2', \eta_1, \eta_2) = {} & \eta_1'^2 + \eta_2'^2 + \eta_1^2\left[\overline{\lambda_0} - 3 + 4B - \frac{\overline{\lambda_\alpha}\ell_0}{b}\right] \\ & + \eta_2^2\left[\overline{\lambda_\alpha} - B\right] \\ & + \eta_2\left[2f + 2b\overline{\lambda_\alpha} - 2bB - \frac{2A\cos\in\sin\theta}{\pi} 2\overline{\lambda_\alpha}\ell_0\right] \qquad \text{-----} \; (13) \\ & + 2fb - Bb^2 - 2b\overline{\lambda_\alpha}\ell_0 - \frac{2bA\cos\in\sin\theta}{\pi} \\ & + O(3) \\ & = h \end{aligned}$$

Where O(3) stands for third and higher order terms in η_1 and η_2 By Liapunov's theorem on stability, it follows that the only criterion for given equilibrium position (0, b) given by (9) to be stable is the v defined by (13) must be positive definite and for this the following three conditions must be satisfied:

$$\left.\begin{array}{ll} (i) & 2f - \dfrac{2A\cos\in\sin\theta}{\pi} + 2b\left(\overline{\lambda_\alpha} - B\right) - 2\overline{\lambda_\alpha}\ell_0 = 0 \\ (ii) & \overline{\lambda_\alpha} - 3 + 4B - \dfrac{\overline{\lambda_\alpha}\ell_0}{b} > 0 \\ (iii) & \overline{\lambda_\alpha} - B > 0 \end{array}\right\} \quad \text{------------ (14)}$$

We have from (9) the y – coordinate of the equilibrium point

$$b = \frac{\dfrac{A\cos\in\sin\theta}{\pi} + \overline{\lambda_\alpha}\ell_0 - f}{\overline{\lambda_\alpha}} \quad \text{------------ (15)}$$

We also have

$$A\cos\in\sin\theta - f > 0 \quad \text{------------ (16)}$$

Using (15) in condition (i) of (14), we find that the

Condition (i) of (14) is satisfied.

Since b is positive so $\overline{\lambda_\alpha} - B$ is also positive

Hence condition (iii) of (14) is also satisfied.

Condition (ii) of 14 and (iii) can be seen as

$$= \overline{\lambda_\alpha} - B + 5B - 3\frac{\overline{\lambda_\alpha}\ell_0}{b}$$

$$-\frac{b\left(\overline{\lambda_\alpha} - B\right) + 5B - 3 - \overline{\lambda_\alpha}\ell_0}{b}$$

$$= \frac{\dfrac{A\cos\in\sin\theta}{\pi} - f + 5B - 3}{b}$$

$$= +ve \ \ if \quad 5B - 3 > 0$$

$$ie \ \ B > \frac{3}{5}$$

Hence all the three condition of (14) are satisfied if

$$B > \frac{3}{5}$$

Thus, equilibrium point given by (9) is stable in the sense of Liapunove if $B > \dfrac{3}{5}$

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