Forbidden 2-colored and 3-colored Posets of Cover-incomparable Line graphs

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Abstract—The cover-incomparability graph of a poset P is the edge-union of the covering and the incomparability graph of P. As a continuation of the study of 2-colored and 3-colored diagrams we characterize some forbidden ⊲ - preserving subposets of the posets whose cover-incomparability graphs are not line graphs is proved.

Key words—Cover-incomparabilitygraph, Blockgraph, Line graph, Poset.

I. INTRODUCTION AND PRELIMINARIES

Cover-incomparability graphs of posets, or shortly C-I graphs, were introduced in [2] as the underlying graphs of the standard interval function or transit function on posets (for more on transit functions in discrete structures [3, 4, 5, 6, 11]). On the other hand, C-I graphs can be defined as the edge-union of the covering and incomparability graph of a poset; in fact, they present the only non-trivial way to obtain an associated graph as unions and/or intersections of the edge sets of the three standard associated graphs (i.e. covering, comparability and incomparability graph). In the paper that followed [9], it was shown that the complexity of recognizing whether a given graph is the C-I graph of some poset is in general NP-complete. In [1] the problem was investigated for the classes of split graphs and block graphs, and the C-I graphs within these two classes of graphs were characterized. This resulted in a linear-time recognition algorithms for C-I block and C-I split graphs. It was also shown in [1] that whenever a C-I graph is a chordal graph, it is necessarily an interval graph, however a structural characterization of C-I interval graphs (and thus C-I chordal graphs) is still open. C-I distance-hereditary graphs have been characterized and shown to be efficiently recognizable [10].

Let P = (V;_≤) be a poset. If $u \leq v$ but $u \neq v$, then we write $u < v$. For u, v ϵ V we say that v *covers* u in P if $u < v$ and there is no w in V with $u < w < v$. If $u \leq v$ we will sometimes say that u is *below* v, and that v is *above* u. Also, we will write $u \lhd v$ if v covers u; and $u \lhd\lhd v$ if u is below v but not covered by v. By $u \parallel v$ we denote that u and v are incomparable. Let V' be a nonempty subset of V. Then there is a natural poset Q = $(V';\leq ')$, where $u \leq ' v$ if and only if $u \leq v$ for any u,v $\epsilon V'$. The poset Q is called a *subposet* of P and its notation is simplified to Q = $(V';\leq)$. If, in addition, together with any two comparable elements u and v of Q, a chain of shortest length between u and v of P is also in Q, we say that Q is an isometric subposet of P. Recall that a poset P is *dual* to a poset Q if for any x, y ϵ P the following holds: $x \leq y$ in P if and only if $y \leq x$ in Q. Given a poset P, its cover-incomparability graph G_P has V as its vertex set, and uv is an edge of G_P if $u \lhd v$, $v \lhd u$, or u and v are incomparable. A graph that is a cover-incomparability graph of some poset P will be called a C-I graph.

Lemma 1 [2**]** Let P be a poset and G_P its C-I graph. Then

(i) G_P is connected;
(ii) vertices in an independent set of G_P lie on a common chain of P;
(iii) an antichain of P corresponds to a complete subgraph in G_P ;
(iv) G_P contains no induced cycles of length greater than 4.

II. 2-colored and 3-colored diagrams

2-coloured diagram $\mathcal{P}$; in [12] we describe the family $\mathcal{P}$ by the Hasse diagram of initial poset P using normal edges, added by the bold edges between u_i and v_j (u_i and v_j are incomparable pairs) for all i and j. It follows that if there is a bold edge between an incomparable pair of elements u_i and v_j in P then either $u_i \lhd v_j$ or $v_j \lhd u_i$,which neither affect the covering nor the incomparability relation of any other pair of elements in P. Any subset of the set of bold edges can thus be chosen and removed arbitrarily to obtain one of the Hasse diagram of a poset from the family $\mathcal{P}$. Hence one drawing, using normal and bold edges, suffices to describe all posets of $\mathcal{P}$.

A 3-coloured diagram Q in [13] is explained as follows. Let G be a C-I graph and H be an induced subgraph of G. We note that there can be different ⊲- preserving subposets Q_i of some posets with G_{Q_i} isomorphic to the subgraph H. Let u,v,w be an induced path in the direction from u to v in H. There are four possibilities in which u, v and w can be related in the ⊲ - preserving subposets. It is possible to have $u \lhd v$, $u \parallel v$, $v \lhd w$ and $v \parallel w$. Each case will appear as a ⊲ - preserving subposet of four different posets. If $u \lhd v$ and $v \lhd w$ in a subposet, then $u \lhd v \lhd w$ is a chain in the subposet and u,v,w is an induced path in H. If there is either $u \parallel v$ or $v \parallel w$ in a subposet Q, then there should be another chain from u to w in Q in order to have u, v,w an induced path in H. We try to capture this situation using the idea of 3-colored diagram. Suppose in ⊲ - preserving subposet Q of a poset P, there exists two elements u,v which is always connected by some chain of length three in Q. Let w be an element in Q such that either both uw and vw are red edges or any one of them is a red edge. Then in order to have a chain between u and v, there must exist an element x in Q so that u, x, v form a chain in Q. When both edges are normal, then we have the chain u,w, v in Q and hence the chain u, x, v is not required in this case. We denote the chain u, x, v by dashed lines between ux and xv in order to specify that it is possible to have the presence or absence of the chain u, x, v in Q. The presence of the chain u, x, v implies that either both of the edges uw and wv are red edges or one of them is a red edge. The absence of the chain implies that both uw and vw are normal edges in Q. We call posets having the above mentioned diagrams as 3-colored diagrams. All subposets of the poset P that we consider in this paper are 3-colored diagrams. Thus by a single 3-colored diagram, we represent a collection of ⊲ - preserving subposets to be forbidden for a poset. In a similar way the dual of a 3-colored diagram is also meaningful and represents a collection of ⊲- preserving dual subposets.

Theorem 2: (Theorem 1,[8]): Let G be a class of graphs with a forbidden induced subgraphs characterization. Let $\mathcal{P}$ = {P | P is a poset with $G_{T_P} \epsilon$ G }. Then $\mathcal{P}$ has a characterization by forbidden ⊲ - preserving subposets.

Theorem 3: (Theorem 7.1.8, [7]) Let G be a graph. Then G is a line graph if and only if G contains none of the nine forbidden graphs of Figure 1 as an induced subgraph.

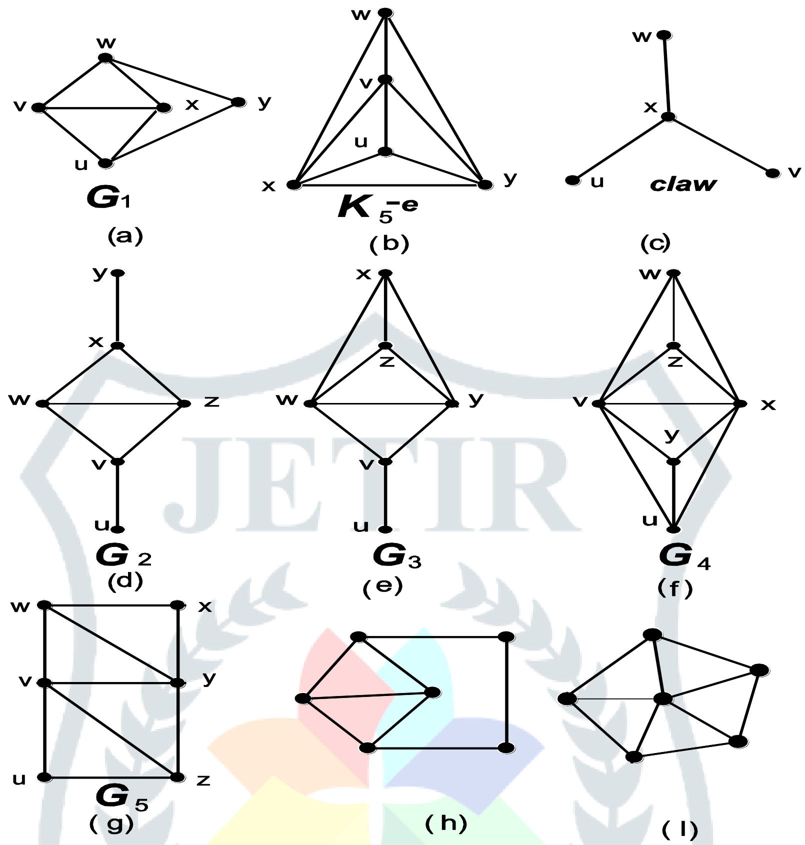
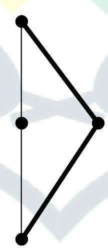


FIGURE 1: NINE FORBIDDEN INDUCED SUB GRAPHS OF LINE GRAPHS

Theorem 4: (Theorem 2.1,[1]) If P is a poset, then G_P is a block graph if and only if P has no 2-colored diagram M_3.



M_3

Figure 2: Forbidden 2-colored diagram for block graph

We consider 2-colored and 3-colored subposets to be forbidden so that its C-I graphs belong to the graph family ℱ**G_4)** of G_4 in Figure1

III.2-Colored and 3-colored ⊲ - preserving subposets of posets whose C-I graphs belong to the family ℱG_4)

We have the following theorem regarding the graph family ℱ**G_4)**

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Theorem 5: If P is a poset, then G_P belongs to ℱ**G_4)** if and only if P contains the 3-colored diagrams Q_5 and Q_6 from Figure 3 and 2-colored diagram P_7 from Figure 4.

Proof. Suppose P contains the 3-colored diagrams Q_i, ; i=5,6 and 2-colored diagram P_7. Then clearly G_P contains the graph from Figure 1(f) as an induced subgraph.

Conversely, suppose $G_P \epsilon$ ℱ**G_4)**. Then G_P contains an induced subgraph G_4 shown in Figure 1(f), with vertices labeled by u, v,w, x, y and z. The vertices u, v, x, w and x, y, z, v induce a diamond in G_4 and since both diamonds are identical, without loss of generality, we consider the diamond induced by u, v,w, x in G_4. By Theorem 4, the vertices u, v,w, x correspond to the 2-colored diagram M_3 as shown in Figure 2. Now we consider the vertices y and z in the graph G_4 and identify all the possible cases in which they can appear as additional vertices in the 2-colored diagram M_3 and obtain the corresponding 3-colored diagrams. Note that y and z are symmetrical pairs in the graph G_4. Thus y and z also occur as symmetrical pairs in the poset also. We first consider y and check all the cases in which y appear in the poset P and then similarly consider the vertex z. We have the following cases. $u \lhd y$ ($y \not\lhd u$ as vy is an edge) or $u \parallel y$.

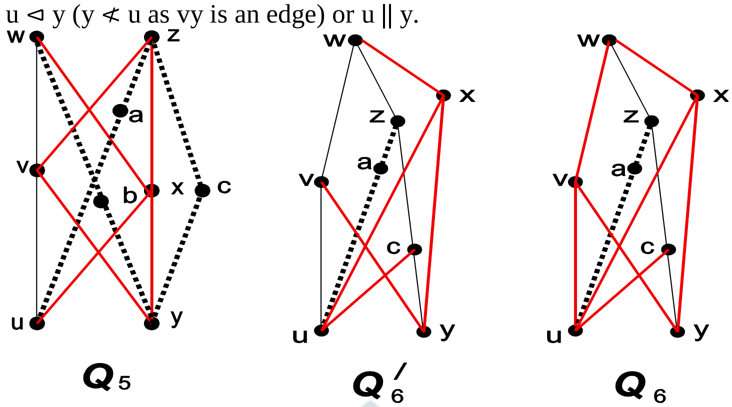


Figure 3: Forbidden 3-colored diagrams for posets whose C-I graphs contains G_4, depicted in Figure 1(f).

Case (1): $u \parallel y$.

Since there is a path of length 2 from y to w in G_4, there must be chains from y to w in P. If $y \lhd v$ ($v \not\lhd y$, since u and y are adjacent in G_4) and $y \lhd x$ ($x \not\lhd y$, since w and x are adjacent in G_4), then $y \lhd v \lhd w$ and $y \lhd x \lhd w$ form corresponding chains. Otherwise, (when both $v \parallel y$ and $y \parallel x$) there must be a dashed line between y and w through b. Hence yv and yx are red edges. A similar case arise if $w \parallel z$, since y and z are symmetric vertices. So the edges vz and zx are red edges and there is a dashed line between u and z. Now consider y and z. Since there is a path of length 2 from y to z in G_4, there must be a chain of length 3 from y to z in P. If $y \lhd x \lhd z$ then we are done. Otherwise, there must be a dashed line between y and z through some point c. In this case, we obtain the 3-colored diagram Q_5 in Figure 3 . If $z \lhd w$, since there is a path of length 2 from y to z in G_4, there must be a chain from y to z in P. Let $y \lhd c \lhd z$ be a chain between y and z. The path from u to z in G_4 is of length 2. Therefore there must be some chain from u to z. If $u \lhd c$, then we are done as $u \lhd c \lhd z$ is the required chain. Otherwise, if $u \parallel c$, there must be a dashed line between u and z. In this case we obtain the 3-colored poset Q'_6 in Figure 3 . In Q'_6, since u and v (v and w) are incomparable, we obtain the same C-I graph, if the normal edges uv and vw are replaced by red edges. Thus we obtain the 3-colored diagram Q_6 in Figure **3.**

Case (2): $u \lhd y$. We consider two sub cases. That is, $w \parallel z$ and $z \lhd w$ ($w \lhd z$ cannot happen as v and z are adjacent).

Sub case (2.1): $w \parallel z$. This case is similar to the case(i) when $z \lhd w$ (viz. $u \parallel y$ and $z \lhd w$). Thus we obtain using similar arguments as case (i), when $z \lhd w$, we obtain the 3-colored poset isomorphic to the dual of Q_6.

Sub case (2.2): $z \lhd w$.

In this case, since there is a path of length 2 from y to z in G_4, there must be a chain between y and z through some vertex c. The only possibility among u, y, z and w is that $u \lhd y \lhd c \lhd z \lhd w$. Thus we obtain the poset P'_7. It may be noted that in this case, the vertices u, v can be in a covering relation or incomparable. Similar is the case with v and w. Since we get the same C-I graph in all the four cases (that is, $u \lhd v$, $u \parallel v$, $v \lhd w$ and $v \parallel w$), we can replace normal edges uv and vw by bold edges. The above conditions can be easily verified and we obtain the2-colored diagram P_7 shown in Figure 4

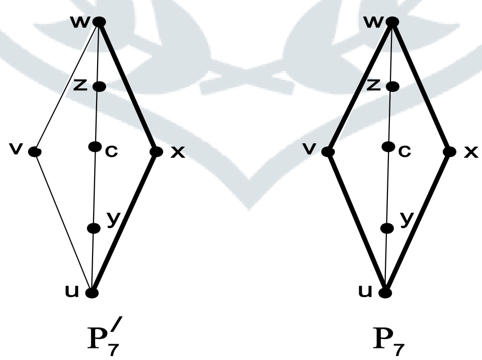
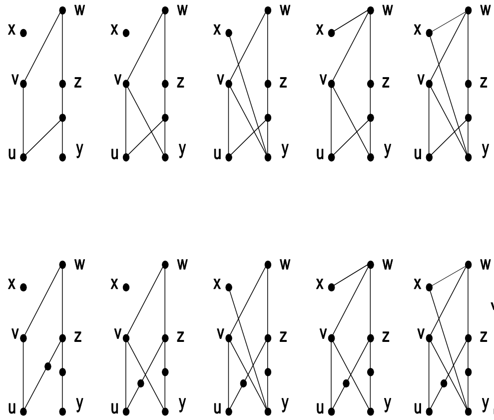


Figure 4: Forbidden 2-colored diagrams for posets whose C-I graphs containsG_4, depicted in Figure 1(f).



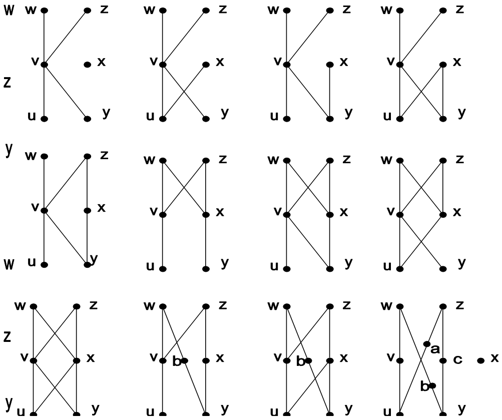


Figure 5:⊲ - preserving subposets corresponding to Q_6 **Figure 6:**⊲ - preserving subposets corresponding to Q_5

Remarks

The number of forbidden ⊲ - preserving subposets of a poset P is such that its C-I graph G_P belongs to a graph possessing a forbidden induced subgraph characterization as instances of the Theorem 2 is in general very large compared to the number of forbidden induced subgraphs. Here we characterize forbidden ⊲ - preserving subposets of G_4 in Figure1 and introduce the idea of 2-colored and 3-colored diagrams to minimize the list of subposets.

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