ORDINAL LATTICE

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Abstract: It introduces notion of ordinal quotients and the structures of partially ordered ordinal numbers.

Keywords : *HCLF, LCRF, Quotient Formation, Lattice ordered binoid (lobinoid).*

INTRODUCTION

Before introducing lattice application to ordinals. We trace out a brief historical development of such numbers.

The foundations of modern mathematics were conceived through a number of discussions between two mathematicians: Cantor and Dedekind during 1872 when they shared their honeymoon in the Harz Mountains. These meetings marked the beginning of a fascinating correspondence in which we can see the birth of set theory.

Cantor indexes the steps of his construction of ordinals by new objects, the transfinite numbers which can be visualized as follows [5, 6, 7, 9].

$0, 1, 2, ..., \omega, \omega+1, ..., \omega+\omega, \ldots, \omega+\omega+\omega, ..., \omega^2, \ldots \omega^3, \ldots$. But this possesses a problem: how far do we have to go. In fact, Cantor found it very hard to define transfinite number, even using the brand-new-and controversial – vocabulary of set theory. But let us follow his reasoning:

Cantor proceeded by Induction, but an induction made transfinite by diagonalization, i.e., it can be continued beyond integers. So it is natural to study the sets satisfying the property underlying the principle of induction. Cantor called them well–ordered sets. To be precise, a well ordered set is a set equipped with an ordering (i.e., its elements can be compared) such that every non-empty subset has a least element. For instance, the set of natural numbers is well ordered. In Contrast, the set of positive reals equipped with natural ordering, is not because, as an example, the set of all reals greater than 1 has no smallest element.

Cantor was then able to define a transfinite number, which is now often called an ordinal. In Naïve set theory [10] it is a well order-type. More explicitely: every well ordered set defines an ordinal, and two of them define the same ordinal if there is an increasing bijection between them-in other words, if they are identical as well ordered sets. In our terminology, an ordinal can be regarded as an equivalence class of well–ordered sets, for the relation defined on the set of well-ordered sets by "there exists an increasing bijection". For instance, the first infinite ordinal numbers, ω, is the class – or order-type – of the set N of natural numbers. Its successor is the order type of the set $N \cup \{\infty\}$where '∞' is a new point that is declared to be greater than any integer; we can also choose the set E_1, to represent $\omega+1$, the set E_2, to $\omega^2 + l$ and the set E_ω to $\omega^\omega + l$. This definition hardly looks Naïve. In fact the ordinals would raise some disturbing existential questions [6, 24, 25].

Cantor regarded his transfinite numbers as generalized integers, whose existence was no less certain than that of the usual integers. He called them unendliche reale ganze Zahlen, and, reale here means 'real' in the sense of reality and not 'real' in contrast to 'complex' (which would be reelle). In a very natural way, he would then define and study a whole arithmetic of his transfinite numbers which extends ordinary arithmetic but does not keep all its properties: The arithmetic of transfinite numbers 'is sometimes useful, the major role is played not by their arithmetic but by their order structure.

Cantor, who has no formal set theory available, relied on his intuition of the integers to produce his transfinite number step by step. As we know that integers never come to an end–you can always 'add 1'. Cantor proceeded to construct the ordinals in the same way but incorporated his idea of diagonalization. He asserted that when there exists an increasing sequence of ordinals but have already been constructed, there exists a smallest ordinal greater than all the ordinals of the sequence - or, if we prefer, an ordinal that is the limit of the sequence. That is how ω, for instance is generated from the sequence of integers. Cantor called his two methods (adding I and taking a limit) the first and second generalizing principle. The second principle is less intutive than the first, and doubters are entitled to ask whether it is legitimate. Cantor felt such doubts himself but could respond only by invoking some kind of psychological obviousness before resorting to the metaphysical arguments of a depressive mind. However, it is inarguable that two sets A and B will have the same number of elements, that is what we call constructing a bijection. To count the elements of an infinite set, it is very tempting to take transfinite numbers as a reference set. But this natural choice leads to at least two problems: that the result of counting a finite set is obviously independent of the order in which its elements were counted. This is not only obvious but perfectly true. But everything changes when we wanted to enumerate an infinite set by means of the ordinals. Imagine, for instance, that we wanted to enumerate the set N of natural numbers. If we do it in the obvious way, by

$\{0, 1, 2, \ldots, n, \ldots\}$

then we assign it the ordinal ω. But we could also start with 1 (as do many of our Anglo-Saxon Contemporaries for whom 0 is not a natural number), then count all the integers greater than 1 and finally remember 0 and count it at the end. Doing so gives the enumeration [7, 9, 24]:

$N = \{1, 2, 3, \ldots, n, \ldots, 0\}$

and we have to assign N the ordinal $\omega + l$. It is not hard to complicate things. If we enumerate all the even numbers first, then all the odd numbers, we assign the ordinal $\omega+\omega$ to N. In fact, we can obtain any ordinal with a representative that is well-ordering on integers. But then how many elements does N have?

Cantor was well aware of the first problem and answered it clearly. He drew a distinction between the number of elements of a set (Machtigkeit or 'power') and the enumeration (Anzahlen) that one could make, a crucial distinction when dealing with infinite sets [6, 19, 20]. Two sets A and B have the same power.

Whenever there exists a one-to-one correspondence between them, although their enumerations may represent different transfinite numbers, as we just saw in the example of N. Moreover, one of the sets may have no obvious well ordering, so that it can't be assigned a transfinite number at first glance. For example, Cantor established a bijection between the set N of natural numbers and the set Q of rational numbers, but this enumeration is not increasing with respect to the obvious ordering: Q is not well – ordered with respect to the usual ordering, because the set of positive rationals has no least element. Similar argument holds for the bijection between the set of algebraic numbers and N. In other words, the set Q can be well-ordered by an ordering that is obtained through a bijection with N and is not the usual ordering. So is there a simple way to define the number of elements in a well orderable set? Cantor gave an affirmative answer to this question by observing that among all the transfinite numbers that can be associated with such a set, there exists a smallest one with respect to the natural well ordering. All that remains is to call this smallest transfinite number the cardinal number or cardinality of the set. To denote these infinite cardinal number, Cantor used the first letter of the Hebrew alphabet, aleph.

This choice is undoubetedly not unrelated to the metaphysical emotions by which he was then tormented nor to his certainty of having opened a new and radical path.

Now we can talk about the cardinality of N. Among all the ways of enumerating N, the smallest (or the simplest, which amounts to the same thing) is of course the first:

N = {0, 1, 2, …, n, …}

Which assigns it the ordinal (.So this ordinal is an aleph, the smallest of them all: we denote it by $\omega = \aleph_0$.

But do there exist infinite sets that are not countable? Cantor had asked this question as early as 1873, in a letter to Dedekind before quickly reaching the surprise conclusion that there does not exists any bijection between the integers and the set of points on the real line. In other words, there are more points on the line than there are integers. In contrast, Cantor would show four years later that there are not more points in the plane or in space, than on the line; i.e., the plane and the space can be put into one-to-one correspondence with the line. These were such novel results that they took their inventor himself by surprise: he would write to Dedekind (in French) on June 29, 1877: "I see it but I don't believe it"…………[9].

To show that there is no bijection between the integers and the real line. Cantor used an argument by contradiction. We now present an abstruct version, which has now become classical, of Cantor's argument. Once the real numbers (or the points of the line) are put into bijection with the set P(N) of subsets of N, it suffices for the proof of the Cantor's theorem to show that there is no bijection between N and P(N). The last statement is just a special case of a very general result : if E is an arbitary set ,there does not exists any bijection between E and P(E).To see this, let f denote an arbitary map from E to P(E). We define a diagonal subset X of E as follows : an element x of E is in X if x is not in the set f(x). Thus subset X of E thus defined is not of the form f(t) for, suppose X = f(t); if t is in X, then t is in f(t), so t is not in X, because X contains only those elements x that are not in f(x). But this is a contradiction. On the other hand if t is not in X, then t is not in f(t), so t is in X because X contains all the elements x that are not in f(x) again,we have a contradiction .Since X can't be written as f(t), we have found a subset of E that is not in the image of map f, so f is not a bijection. This completes the proof.

The proof above is very simple relative to the complexity of contemporary mathematics, but it is interesting for more than one reason. First it shows that P(E) has more elements than E ,with no restrictions on the set E. So we can repeat the argument to obtain larger and larger sets: E, P(E), P(P(E)), and it keeps going, but where does it stop? Well it keeps going as long as there are ordinals, and we are led back to the trickly question we ran into earlier. This is the simplified approach.

We have just shown that the set of points on the line has more elements than the set of integers, but we have not yet mentioned its cardinality. This leads us to the second question raised by the counting of infinite sets. We know how to associate a cardinal with any well –orderable set; conversely a set must be well orderable for us to be able to discuss its cardinality because its cardinality is the simplest of its well ordering. In other words, we can speak of cardinality of sets whose elements can in some sense be considered - counted – one after another. But are all sets orderable? Is the set for reals for instance? It would have to be if we wanted to talk about its cardinality. For that matter, does there exists at least one well – orderable uncountable set?

Cantor gave an affirmative answer to the last question by showing that the set of all countable transfinite number is not countable, even though it is well ordered. It is uncountable simply, because, by definition, it contains all the countable well ordered sets. If it were itself countable, then because it is well ordered there would be an increasing bijection between it and one of its subsets, the one consisting of all the ordinals less than some countable ordinal α. But to set up an increasing bijection between two well ordered sets, we have to pair the elements successively, starting with the smallest and not skipping any. If we proceed in this way, we will match all the ordinals smaller than α with themselves, but it will be impossible to continue the construction up to α and beyond. So if we enumerate all the countable ordinals:

$0, 1, 2, \ldots, \omega, \omega+1, \ldots, 2\omega, \ldots, \omega^2, \ldots, \omega^3 \ldots, \omega^\omega \ldots, {}_\omega\varepsilon_0, \ldots$

we eventually obtain an uncountable well ordered set. Cantor presented this uncountable cardinal in his Grundlagen in 1883. 1t is clearly the smallest uncountable cardinal he called it alephl, and denoted it by $_I$. Of course, for him this uncountable cardinal was the cardinality, of the real line, which is called the power of countinuum and denoted by 2, because it is power set of the set of subsets of the integer. But we need to exhibit a bijection between set of countable ordinals and the set of reals. This is the countinuum problem, which can be summarized as follows: Is the equation $2\aleph^O = \aleph_1$ true?

Cantor was to expend enormous effort-to the point of nervous breakdowns-trying to obtain the affirmative answer that would have been crowning, achievement of his theory. He believed in it until the end. We know today that, despite all his talent, it was impossible for him to succeed. Godefroy [9] proved in 1940 that the equation could not be proved to be false in standard set theory (which had meanwhile been axiomatised); [10] proved in 1963 that, in the same theory it could not be proved to be true.

This equation now called the continuum hypothesis [11], is therefore undecidable. The problem of 'constructively' well-ordering an uncountable well-ordered set is the most fundamental problem of the set theory [11, 12, 13]. It should be noted that the class of all ordinals though well-ordered is not itself an ordinal because it is not a set [1, p 203]

Von Neumann [12, 14] defined an ordinal number as a well-ordered set α such that

$S(\xi) = \xi$, for all ξ in α, where

$S(\xi) = \{\eta \ \varepsilon\ \alpha : \eta < \xi\}$ i.e. $0 = \varphi$, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$….

$n+1 = \{0,1,2, \ldots, n\}\ldots$, $\omega = \{0, 1, 2, \ldots\}$, $\omega = \omega \cup \{\varepsilon\}$

He considered ω as an ordinal of second kind and is the greatest ordinal all whose members are ordinals of the first kind. The symbol ε_o is the least upper bound of the sequence: $1, \omega, \omega^\omega, \omega^\omega, \ldots$ and the symbol Ω to represent a well ordered set consisting of all the ordinals of the first and the second kinds. In von Neumann set theory the class of all ordinals does exists, but it is a proper class and thus can't be a member of itself or any other class.

R.M. Robinson, P. Bernays and K. Godel and others [9, 13, 14, 15. 16] introduced an independent theory of ordinals without referring the definition to the concept of order.

R.N. Lal [17] gave a complete extention of ordinals with their non-commutative operations. The difficulty what he felt in density problem due to non-commutativity was resolved through the introduction of ordinal continued fraction. His concept of fraction is quite different from that of E. Zakon [18]. He introduced ordinal rationals with their incompleteness and ordinals reals with their completeness in detail.

1. OUOTIENT FORMATION

Only a few mathematician have worked on lattices infused with ordinals.some of them [17, 18, 19] discovered some conditions under which chains can be embedded into ordinals. O. Bernard [20] invested regular contexts of ordinal sums and ordinal products of two lattices.

Our endeavour is to build up a theory of ordinal lattices which emerges from consideration of extention of the relation of divisibility to ordinals. The left cancellation law on ordinals leads to the concept of left divisibility ,the highest common left factor (hclf (Λ)) and dually the least common right multiple (lcrm (v)) which induces primeness of ordinals and enriches the ordinal number theory.

Definition 2.1: *Let $\alpha \neq 0'$ and β be two ordinals .Then we say that α divides β (in symbol, α/β) iff there exists an ordinal γ such that $\beta=\alpha*\gamma$. Here γ is called the left quotient of β by α and is designated as $[\beta: \alpha]$. A non-zero ordinal γ is called:*

(I) The hclf of α and β (ie, $\gamma = \alpha\Lambda\beta$) iff γ is the largest left divisor of both α and β, i.e iff $\gamma \mid \alpha$, $\gamma \mid \beta$ and further $\delta \mid \alpha$ and $\delta \mid \beta$ implies that $\delta \mid \gamma$ for every $\delta \neq 0$.

(II) The lcrm of two non-zero ordinals a and β (i.e, $\gamma = \alpha V\beta$) iff γ is the smallest ordinal such that α and β are left divisors of γ, i.e, iff $\alpha \mid \gamma$, $\gamma \mid \beta$ y and further $\alpha \mid \delta$ and $\beta \mid \delta$ implies that $\gamma \mid \delta$ for every δ.

In the ensuring study we shall assume α as a nonzero ordinal number involved in $[\beta: \alpha]$ for every ordinal β.

Theorem 2.1: The quotient formation [:] has the following properties:

(I) $[\alpha: \gamma] + [\beta: \gamma] = [\alpha+\beta: \gamma]$
(II) $[\beta: \gamma] \quad [\alpha: \beta] = [\alpha: \gamma]$
(III) $[2*\alpha : 2 *\beta] = [\alpha: \beta]$
(IV) $[2* \alpha: \beta] = [\alpha: \beta]$ provided α is a limit ordinal.
(V) $[\alpha : \alpha] = 1$
(VI) $[\alpha: \beta] = [1: [\beta: \alpha]]$
(VII) $[\beta: \alpha]*\gamma = [\gamma: [\beta: \alpha]]=[\beta*\gamma: \alpha]$
(VIII) $[[\alpha: \beta]: \gamma] = [\alpha: \beta*\gamma]$
(IX) $[[\alpha:\gamma] [\beta:\gamma]] = \quad [(\alpha:\beta]$

Proof: (I) Put $\rho =[\alpha:\gamma]$ and $q=[\beta:\gamma]$
Then $a = \gamma * p$ and $\beta = \gamma * q$
$\Rightarrow \alpha + \beta = \gamma * (p+ q)$
$\Rightarrow p + q = \alpha + \beta:\gamma$

(II) Put $p = [\alpha: \beta]$, $q = [\beta: y]$
Then $\alpha = \beta * p$ and $\beta = \gamma * q$
$\Rightarrow \alpha = \gamma * q * p$
$\Rightarrow [\alpha : \gamma] = q * p$

(III) Put $p = [2*\alpha : 2: \beta]$
Then $2 * \alpha = 2 * \beta *p \Rightarrow \alpha = \beta*b$

(IV) The proof is an immediate consequence of the fact that $2 \alpha= \alpha$ when a is a limit ordinal.

(V) Obvious.

(VI) Put $p = [\alpha: \beta]$, $q = [\beta:\alpha]$ and $r = [1:q]$
Then $1= q * r$, $\alpha = \beta * p$, $\beta = \alpha * q$
$\Rightarrow \alpha = \alpha * q * p$
$\Rightarrow 1= q * p$
$\Rightarrow q * r = q * b$
$\Rightarrow p = r$

(VII) Put $[\beta:\alpha]$. Then $\beta= \alpha *\rho$
$\Rightarrow \beta*\gamma = \alpha*p*\gamma$
$= a * (p*\gamma)$
$\Rightarrow p*\gamma = \beta*\gamma*\alpha$

Also, $[\gamma :[\alpha : \beta]] = \xi$ and $[\alpha : \beta] = \eta$
$\Rightarrow \gamma = \eta * \xi$
$\Rightarrow \beta * \gamma = (\beta*\eta) * \xi$
$\Rightarrow \beta * \gamma = (\beta*\eta) * \xi$
$\Rightarrow \beta * \gamma = \alpha * \xi$
$\xi = [\beta * \gamma * \alpha)$

(VIII) Put $[\alpha : \beta] : \gamma] = \xi$
Then $\alpha : \beta= \gamma * \xi$
$\Rightarrow \alpha = \beta * \gamma * \xi$
$\Rightarrow [\alpha : \beta * \gamma] = \xi$

(IX) Put $[\alpha : \gamma] = p$, $[\beta : \gamma] = q$ and $[p : q] = r$
Then $\alpha = \gamma * q$, and $[p : q] = r$
$\Rightarrow \gamma * p = \gamma * (q*r) = (\gamma*q)*r = \beta * r$

$\Rightarrow \alpha = \beta * r$

Remark 2.1: In view of the result (vi) of the above theorem we may write $[\alpha : \beta] = [\beta : \alpha]^{-1} = 1/[\beta : \alpha]$ and that of (IX),

$[\alpha : \gamma] / [\beta : \gamma] = [\alpha : \beta] = \alpha / \beta$

Theorem 2.2: (I) ω is a divisor of every ordinal of the second kind

(II) If α and β are ordinals of the second kind and $\alpha \mid \beta$. Then there exists a pair of ξ ordinals and η such that one of them divides the other η

(III) For any two ordinals α and β,

$(\alpha \wedge \beta) * (\alpha \vee \beta) \leq \beta \leq) (\beta * \alpha) \vee (\alpha * \beta)$

Proof: (I) For, if α is an ordinal number of the second kind, we can obtain an additively indecomposible ordinal $\omega * \xi$ for some ordinal ξ such that $\alpha = \omega * \xi$

(II) For, $\omega \mid \alpha$ and $\omega \mid \beta \Rightarrow \exists$ ordinals ξ and η such that

$\alpha = \omega * \xi$ and $\beta = \omega * \eta$

Further, $\alpha/\beta \Rightarrow \exists$ an ordinal γ such that $\beta = \alpha * \gamma$

$\Rightarrow \omega * \eta = \omega * \gamma$

$\Rightarrow \eta = \xi * \gamma$

(III) for, $\{(\alpha \wedge \beta) * \alpha\} \vee \{(\alpha \wedge \beta) * \beta\}$

$\leq ((\beta * \alpha) \vee (\alpha * \beta)$

The following theorem induces the notion of minimal sequence of ordinals

Theorem 2.3: If α is not minimal in ω, then there exists an ordinal β such that $\beta * 2 \leq \alpha$

Proof: If α is not minimal, $\exists\ \beta_1 \varepsilon\ \omega$ such that $\beta_1 < \alpha$ and hence their exists a unique ordinal $\beta_2 < \alpha$ with $\alpha=\beta_1+\beta_2 \geq \beta + \beta = \beta*2$, where β is defined as the smaller of β_1 and β_2

Definition 2.2: *A minimal sequence (α_i) of elements in ω is one containing but one element α_i which is minimal, or containing a denumerable infinitude of elements such that*

$\alpha_{i+1} *2 \leq \alpha_i$ *(i=1, 2, ...)*

Remark 2.2: There exists a minimal sequence.

Lemma 2.1: If $\alpha\ \varepsilon\ \omega$. Then $\lim\limits_{i\to\infty}[\alpha_i : \alpha] = \infty$ in the case when the minimal sequence is infinite

Proof: We have by the theorem (2.1)(II)

$[\alpha_i:\Upsilon] = [\alpha_{i+1}:\Upsilon] * [\alpha_i:\alpha_{i+1}]$

$\Rightarrow [\alpha_i:\Upsilon] \geq [\alpha_{i+1} : \Upsilon] * 2$

By the induction.

$[\alpha_{i+j}:\Upsilon] \geq [\alpha_{i+1} : \Upsilon] * 2^j$

Putting i=1 and j=k-1, we have

$[a_k, : \Upsilon] \geq [a_2:\Upsilon] * 2^{k-1}$

Therefore $\exists\ i_0$ such that

$[\alpha_i : \Upsilon] \geq 2^{i-1}, \forall\ i \geq i_o$

$\lim\limits_{i\to\infty}[\alpha_i : \alpha] = \infty$

Theorem 2.4: Let $\alpha \neq 0$, $\beta \neq 0$ be given.

Then $\lim\limits_{i\to\infty}\dfrac{[\beta : \alpha_i]}{[\alpha_i : \alpha]}$ exists and is > 0 ,< ∞ (If (α_i) consists of one element A_1 min ,we mean by $\lim\limits_{i\to\infty}$ the value at i = 1

Proof: (I) Let $\alpha_i = \alpha_1$, be minimal. Then $\alpha_I \neq 0$ and $\beta = \alpha_I * [\beta:\alpha_1]$. Now $[\beta:\alpha_1] \neq 0$ since $\beta \neq 0$, and $\alpha = \alpha 1 * [\alpha_i = \alpha_1]$ whence $[\alpha, \alpha_1] \neq 0$

Hence $\lim\limits_{i\to\infty}\dfrac{[\beta : \alpha_i]}{[\alpha_i : \alpha]}$ exists and has the desired property.

(II) Let (α_i) be infinite and minimal. By the Theorem 2.1(II)

$[\beta:\alpha_{1+1}] \leq ([\alpha : \alpha_{i+1}] +1) * [\beta : \alpha_i] +1)$

and $[\alpha : \alpha_{i+1}] = [\alpha i : \alpha_{i+1}] * [\alpha:\alpha_i]$

$\therefore [\beta:\alpha_{i+1}]/[\alpha:\alpha_{i+1}] \leq ([\alpha: \alpha_{i+1}]+1)/[\alpha : \alpha_{i+1}] * [\beta : \alpha_i]+1)/[\alpha:\alpha]$

$\leq (1+(1/2^i)) * ([\beta:\alpha_i]+1)/[\alpha:\alpha_i]$

$$\begin{pmatrix} \because [\alpha_i : \alpha] \geq 2 \\ \Rightarrow \dfrac{1}{[\alpha_i : \alpha_{i+1}]} \leq \dfrac{1}{2} \\ \text{for each } i \end{pmatrix}$$

$$\overline{\lim_{i\to\infty}}\frac{[\beta : \alpha_{i+1}]}{[\alpha_i : \alpha_{i+1}]} \leq \frac{[\beta : \alpha_i]}{[\alpha : \alpha_i]}$$

$$\overline{\lim_{i\to\infty}}\frac{[\beta : \alpha_k]}{[\alpha : \alpha_k]} \leq \frac{[\beta : \alpha_i]}{[\alpha : \alpha_i]}$$

$$\overline{\lim_{i\to\infty}}\frac{[\beta : \alpha_k]}{[\alpha : \alpha_k]} \leq \underline{\lim_{i\to\infty}}\frac{[\beta : \alpha_i]}{[\alpha : \alpha_i]}$$

$$= \underline{\lim_{i\to\infty}}\frac{[\beta : \alpha_i]}{[\alpha : \alpha_i]}$$

Since $1/[\alpha:\alpha i]$ tends to zero by the lemma 2.1 .But this implies the existence of the desired limit .It is finite also, since it is less than or equal to $([\beta:\alpha_i]+1)/[\alpha:\alpha_i]$, and since this is finite if i is sufficiently great . Since the same reasoning applies to the reciprocal $[\alpha:\alpha_i]/[\beta:\alpha_i]$, this fraction also has a finite limit ,whence the limit of $[\beta:\alpha_i]/ [\alpha:\alpha_i]$ exists.

This completes the proof.

Definition 2.3: *If $\alpha \neq 0$, $\beta \neq 0$, we define*

$$(\beta : \alpha) = \lim_{i\to\infty} \frac{[\beta : \alpha_i]}{[\alpha : \alpha_i]}$$

***Theorem 2.5*:** Let α, β, Υ are different from 0.Then the following results hold:

(a) $(\alpha,\alpha) =1$

(b) $(\alpha:\beta) = (\beta:\alpha)^{-1}$

(c) $(\alpha:\Upsilon) = (\beta:\Upsilon) * (\alpha:\beta)$

(d) $(\alpha+\beta:\Upsilon) = (\alpha:\Upsilon) = (\beta:\Upsilon)$

(e) $\alpha > \beta \Rightarrow (\alpha:\Upsilon) > (\beta:\Upsilon)$

***Proof*:** (a) This is obvious, since $[\alpha:\alpha_i]/ [\alpha:\alpha_i] = 1$, for sufficiently large i,

(b) $$(\alpha : \beta) = \lim_{i\to\infty} \frac{[\alpha : \alpha_i]}{[\beta : \alpha_i]}$$

$$= \lim_{i\to\infty} \left(\frac{[\beta : \alpha_i]}{[\alpha : \alpha_i]} \right)^{-1}$$

$$= (\beta : \alpha)^{-1}$$

(c) $$(\alpha : \Upsilon) = \lim_{i\to\infty} \frac{[\alpha : \alpha_i]}{[\Upsilon : \alpha_i]}$$

$$= \lim_{i\to\infty} \frac{[\alpha : \alpha_i]}{[\beta : \alpha_i]} * \lim_{i\to\infty} \frac{[\beta : \alpha_i]}{[\Upsilon : \alpha_i]}$$

$$= (\alpha : \beta) * (\beta : \Upsilon)$$

(d) (I) Let a_l be minimal, and put $þ = [\alpha:\alpha_i]$, $q = [\beta:\alpha_1]$ and $s = [\Upsilon:\alpha_1]$. Then $\alpha = \alpha_1 * þ$, $\beta:\alpha_1 * q$, $\Upsilon = \alpha_1 * s$

$\therefore\ \alpha + \beta = \alpha_1 * (þ+q) \Rightarrow [\alpha+\beta:\alpha_1] = þ+q$

and $(\alpha+\beta:\Upsilon) = [\alpha+\beta:\alpha_1] = þ+q/s = þ/s + q/s = (\alpha : \Upsilon) + (\beta : \Upsilon)$

(II) Let α_i be infinite (i = 1, 2, 3, …) Then by the theorem 4.1(I)

$[\alpha:\alpha_i] + [\beta:\alpha_i] = [\alpha+\beta:\alpha_i]$

If we divide by $[\Upsilon:\alpha_i]$and let i tend to infinity, each term has a limit and from which (d) follows:

(e) There exists β_1 with $\alpha = \beta+\beta_1$ and $\beta_1 \neq 0$

Then $(\alpha : \Upsilon) = (\beta : \Upsilon) + (\beta_1 : \Upsilon) > (\beta : \Upsilon)$

2. LATTICE ORDERED BINOID (LOBINOID)

In this section we shall study the lattice ordered structure of the set of nonzero ordinal numbers: α, β, Υ, …

***Lemma 3.1*:** The relation of divisibility ($\leq$) partially orders the set ω of non-zero ordinal numbers with respect to which the monoid (ω, x) is left ordered. The poset with hclf ($\wedge$) and lcrm ($\vee$) is a lattice.

***Proof*:** (I) ‘That $\leq$ is a porelation and helf and lcrm are lattice operations are obvious.

(II) $\alpha \leq \beta$(i.e. $\alpha \mid \beta$) $\Rightarrow \exists$ an ordinal Υ such that $\beta = \alpha * \Upsilon$

$\Rightarrow \delta * \beta = \delta * (\alpha * \Upsilon)$

$= (\delta * \alpha) * \Upsilon$

$= \delta * \alpha|\delta * \beta$

***Theorem 3.1*:** The set $(\omega,+,*)$ is a lobinoid [21,22]

***Proof*:** $(\omega, +, *)$ equipped with associative binary operation is a bisemigroup with the property : $n+\omega = n\omega=\omega$ for every ordinal n of the first kind. Thus the existence of a common natural element ω for both the associative binary operations together with the lemma 3.1 yield the proof of the theorem.

Definition 3.1: *An ordinal number α is said to be even iff 2 \ α; Odd, otherwise prime iff for every ordinal number Υ either $\alpha \mid \Upsilon$ or hclf (α Υ)=1.Two ordinals α and β are said to be co-prime iff hclf (α,β) = 1*

Example 3.1 An initial transfinite ordinal number ω is even where $\omega+1$ is odd and they are prime .Further Ω is prime.

***Solution*:** Since $\omega=2* \omega$ and $\omega+1 = 2* \omega+1$

***Remark 3.1*:** Since every ordinal can be represented either as $2*\alpha$ or as $2*\alpha+1$, each ordinal number is either even or odd ,for example

$(\omega+1) *2 = 2* (\omega+2)+1$ is odd .

The Fermat's factorization method (i.e., an odd number can be factorized iff it is a difference of two squares [21, 22, 23, 24]) fails for ordinals, since

$\omega^{n+1}= \omega^n * \omega = 2*\omega^n*\omega$ is an even ordinal and

$\omega^{n+1} = (\omega^n+\omega)^2 – (\omega^n)^2$

Again ω^2 has infinitely many such representations:

$\omega^2=[(\omega* (n+1)]^2 — [\omega*n]^2$, for n = 1, 2, …

Moreover, Goldback hypothesis (i.e., every even natural number > 2 is the sum of two prime number [25]) is also false for ordinals, for w+10 is not such a sum.

We have the following extension of the familiar factorization theorem for finite ordinals :

***Theorem 3.2*: Every ordinal α >1which is not itself a prime, is a product of a finite number of primes, sometimes in more than one -way.

***Proof*: There is a least ordinal β>1 such that $\alpha = \alpha_1 * \beta$, where $1<\alpha_1<\alpha$. If α_l is not a prime, the argument may be repeated on α_1. Since a decreasing sequence of ordinals must be finite, we have the desired result.

As an example of non-uniqueness we may write:

$$\begin{aligned} \omega^2 &= (\omega+1) * \omega \\ &= (\omega+1) * 2 * \omega \\ &= (\omega+1) * 3 * 2 * \omega \\ &= 5 * (\omega+1) * 7 * \omega \text{ etc.} \end{aligned}$$

One of the most important concepts in ω is that of residual. Historically speaking it draws primary inspiration for the work M. Ward and R.P. Dilworth appearing in a series of important papers [3, 23, 24, 25]. Our study is naturally base on non-commutative case. The right residual $\alpha : \beta$ of α by β is the largest Υ such that $\beta * \Upsilon \le \alpha$. A right residuated lattice is an I-groupoid (Lattice-groupoid) in which right residual for every pair of elements exists.

***Theorem 3.3*: ω is right residuated.

***Proof*: For ordinals α and β in ω there exists a unique ordinal Υ and a unique ordinal 6< a in such that

$$\alpha = \beta * \Upsilon + \delta$$
$$\Rightarrow \beta * \Upsilon \le \alpha$$

***Remark 3.2*: From the lemma 3.1 and the theorem 3.1, 3.2 and 3.3 it follow that ω is a right residuated lobinoid.

4. REFERENCES

[1] G. Birkhoff : Lattice Theory, AMS coll. Publication, vol. 25, Providence, Rhode Island, 1984.
[2] R.P. Dilworth : Non-commutative residuated Iattices, Trans Amer. Math. Soc., 46, 1939, 426-444.
[3] R.P. Dilworth and M. Ward : Residuated Lattices, Trans Amer. Math. Soc., 45, 1939, 335-354.
[4] M. Ward : Residuated distributive lattices, Duke Math. J. 6, 1940, 641-651.
[5] G. Canton : Contributions to the founding of the theory of Transfinite numbers, Translated by E.B. Joundain, Chicago, 1915, Reprinted by Dover Pub. INC.
[6] ----------- : Gesammelte mathematischen and Philosophischen Inhalts, Edited by Zermelo, Springer, Berlin, 1932.
[7] J.S. Jha, A.K. Mishra & Shwetambra : Anusandhan, vol 10, No. 16, 2008, 73-78.
[8] J.S. Jha, M.K. Singh & B.P. Kumar : Proc. of Bihar Math. Soc., 2008.
[9] G. Godefroy : From the paradise that Cantor has created for us. The Adventure of Numbers, AMS, vol. 21, 2004, 97-111.
[10] P.R. Halmos : Naïve set theory, East-West Press, New Delhi 1972 (P. 75).
[11] K. Godel : The consistency of continuum Hypothesis, Princeton Univ. Press, 1940 (P23-25).
[12] Y. Maschovakis : Notes on set theory, UCLA, Mathematics Department, Los Angles, CA, USA, second Ed, 2006, Springer-verlag.
[13] P. Komjath and V. Totik : Problems and theorems in classical set theory, Spriner-Verlag, 2006.
[14] Von Neumann : Zur E-infuhrung der transfiniten Zahlen, Aeta Litterarum ac Scientiarum, Sectro Scientiarum Mathmaticarum, Tom 1, 1922-23, p190-208.
[15] R.M. Robinson : Theory of classes. J. of Symbo. logic, 2, 1937, 29-36.
[16] P. Bernays : A System of Axiomatic Set theory. J. of Symb. Logic, 6, 1941, 6-12.
[17] R.N. Lal: A complete extension of ordinal number. AMS, Vol. 70, No.5, May 1963.
[18] E. Zakon : On fractions of ordinal numbers, Israel Institute of Technology (Technion) Sci, Pub, 6, 1955, 94-103.
[19] S. Kuhlmann : Isomorphism of Lexicographic power of reals. Proc. Of AMS, 123, No. 9, 1995, 2657-62.
[20] O.Bernard : On context of direct products, ordinal sums and ordinal products (English Summary). Acta. Univ. Corolian, Math., Phy., 44 (2003), no. 2, 3-15.
[21] W.B.V. Kandasamy : Bialgebraic Structures and Smarandache Bialgebraic Structures Amer Research Press, Rehoboth, NM, 2003.
[22] ----------- : Bisemigroups and its Applications. J. of Bihar Mathematical Society, Vol 23, 2003, 35-44.
[23] I. Kleiner : Fermat: The founder of Modern Number Theory, Math. Mag. Vol 78, No.1, 2005, 3-14.
[24] W.A. Coppel : Number Theory, Springer-Verlag, 2006 (An introduction to Mathematics, Part A)
[25] P. Suppes: Axiometic Set Theory. East-West Press, New Delhi, 1995.

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