HOMOTOPY PERTURBATION METHOD FOR SOLVING LINEAR AND HOMOGENEOUS EQUATIONS

Dr.T.ShanmugaPriya[1] and R.Santhiya[2]
Asst. Professor[1], PG Research[2]
Department of Mathematics
[1] & [2] - Adhiyaman Arts and Science College For Women, Uthangarai, Tamil Nadu

Abstract: Although attempts have been made to solve time-dependent differential equations using homotopy perturbation method (HPM), none of the researchers have provided a universal homotopy equation. In this paper, going one step forward, we intend to make some guidelines for beginners who want to use the homotopy perturbation technique for solving their equations. These guidelines are based on the L part of the homotopy equation and the initial guess. Afterwards, for solving time-dependent differential equations, we suggest a universal L and v0 in the homotopy equation. Examples assuring the efficiency and convenience of the suggested homotopy equation are comparatively presented.

Keywords: Homotopy Perturbation Method, linear equation, homogeneous equation, Evolution equations, Cauchy reaction-diffusion equations, Klein-Gorden equation.

1. Introduction

In recent years, the homotopy perturbation method (HPM), first proposed by Dr. Ji Huan He, has successfully been applied to solve many types of linear and nonlinear functional equations. This method, which is a combination of homotopy in topology and classic perturbation techniques, provides us with a convenient way to obtain analytic or approximate solutions for a wide variety of problems arising in different fields. Dr. He used HPM to solve Lighthill equation, Duffing equation and Blasius equation, and then the idea found its way in sciences and has been used to solve nonlinear wave equations, boundary value problems, quadratic Riccati differential equations, integral equations, Klein-Gordon and sine-Gordon equations, initial value problems, Schrödinger equation, Emden-Fowler type equations, nonlinear evolution equations , differential-difference equations, modified KdV equation and many other problems. This wide variety of applications show the power of HPM in solving functional equations (although we know it has limitations, that will be mentioned later on). Studying the method, we understand that the idea is straightforward, but everyone has solved his/her own problem, heuristically, using some tricks.

Although this shows the flexibility of the method, a beginner confronts problems using it. In this paper, we intend to somehow generalize the idea and make some guidelines. These guidelines may have been discovered by other researchers but no one has presented them as general guidelines. After this general discussion, we restrict ourselves to the case of time-dependent differential equations and suggest a quite simple technique for using HPM. Comparatively speaking, even though we don't claim that the suggested technique is the best one, this is a reliable technique which one can simply use without having much experience and understanding of HPM.

2. Basic idea of homotopy perturbation method

For a good understanding of the homotopy perturbation method, the reader is referred to   Dr. He's works [1, 2], where more developments could be found in [21,22]. Also Liao's works [23, 24] would be a good reference for this development, because this method is quite similar to the method proposed by Liao, known as homotopy analysis method (HAM). To describe the basic ideas, consider the time-dependent differential equation in the following general form

$$A\big(Y(r,t)\big) - f(r,t) = 0, \tag{1}$$

Where A is a differential operator, y(r, t) is an unknown function, r and t denote spatial and temporal independent variables, respectively, and f(r, t) is a known analytic function. A, generally speaking, can be divided into two parts, L and N,

$$A = L + N \tag{2}$$

Where L is a simple part which is easy to handle and N contains the remaining parts of A. Using homotopy technique one can construct a homotopy ϕ (r. t; q) satisfying

$$H(\phi(r.t;\ q), q) = (1 - q)\{L\left(\phi(r.t;\ q)\right) - L(v_0(r,t))\} + q\{A\left(\phi(r.t;\ q)\right) - f(r,t)\}\ 0, \tag{3}$$

Where $q \in [0,1]$ is an embedding parameter and $v_0(r,t)$ is an initial guess for Eq. (1) which satisfies initial/boundary condition(s). Eq. (3) is called homotopy equation. Equivalently it can be written as follows:

$$L\left(\phi(r.t;\ q)\right) - L(v_0(r,t)) + q\{A\left(\phi(r.t;\ q)\right) + L(v_0(r,t)) - f(r,t)\} = 0. \tag{4}$$

Clearly we have

$$q = 0 \Rightarrow H(\phi(r.t;\ 0), 0) = L\left(\phi(r.t;\ 0)\right) - L(v_0(r,t)) = 0, \tag{5}$$

$$q = 1 \Rightarrow H(\phi(r.t;\ 1), 1) = L\left(\phi(r.t;\ 1)\right) - L(v_0(r,t)) = 0, \tag{6}$$

Which the latter is actually Eq.(1) with solution y(r, t). Eq. (5) has $v_0(r,t)$ as one of its solutions in the case where L is assumed to be linear, $v_0(r,t)$ is the only solution. So we have

$$\phi(r.t;\ 0) = v_0(r,t),$$
$$\phi(r.t;\ 1) = y(r,t).$$

The changing process of q, from zero to unity, is just that of ϕ (r.t; q) from $v_0(r,t)$ to $y(r,t)$ this is called deformation. If the embedding parameter $q(0 \leq q \leq 1)$ is considered as a ``small parameter'', applying the classic perturbation technique, we can naturally assume that the solution to Eqs. (5) and (6) can be given as a power series in q, i.e.

$$\phi (r.t; q) = u_0(r,t) + u_1(r,t)q + u_2(r,t)q^2 + \cdots. \tag{7}$$

Using (7) for $q = 1$, one has

$$y(r,t) = u_0(r,t) + u_1(r,t) + u_2(r,t) + \cdots \tag{8}$$

which is the approximate solution to Eq. (1) (see, e.g. [1, 2]). In most cases the series (8) is a convergent one which leads to the exact solution of Eq. (1). One can take the closed form or truncate the series for obtaining approximate solutions. As this method is an iterative method, so the Banach's fixed point theorem can be applied for convergence study of the series (8). The interested reader can refer to [25].

3. Guidelines for choosing homotopy equation

In a homotopy equation, What we are mainly concerned about are the auxiliary operator L and the initial guess v_0. Once one chooses these parts, the homotopy equation is completely determined, because the remaining part is actually the original equation and we have less freedom to change it. Here we discuss some general rules that should be noted in choosing L and v_0.

Discussion on L:

According to the steps of the homotopy perturbation procedure, L should be:

(i) ``Easy to handle''.

We mean that it must be chosen in such a way that one has no difficulty in subsequently solving systems of resulting equations. It should be noted that this condition doesn't restrict L to be linear. In some cases, as was done by He in to solve the Lighthill equation, a nonlinear choice of L may be more suitable. But, it's strongly recommended for beginners to take a linear operator as L.

(ii) ``Closely related to the original equation''

Strictly speaking, in constructing L, it's better to use some part of the original equation. We can see the effectiveness of this view in where Chowdhury and Hashim have gained very good results with technically choosing the L part (later on we will show that they could choose other operators as L).

There is no universal technique for choosing the initial guess in iterative methods, but from previous works done on Homotopy Perturbation Method and our own experiences, we can conclude the following facts:

(i) ``It should be obtained from the original equation''.

For example,

It can be chosen to be the solution to some part of the original equation, or it can be chosen from initial/boundary conditions.

(ii) ``It should reduce complexity of the resulting equations''.

Although this condition only can be checked after solving some of the first few equations of the resulting system, this is the criteria that has been used by many authors when they encountered different choices as an initial guess.

4. The classic vive on Homotopy Perturbation Method

As done by many authors, in order to obtain a good approximation, one has to test different choices of L and v_0 and then choose the most suitable of all. This is the classic approach in using Homotopy Perturbation Method and there is no general rule to choose L and v_0. Here we present two comparative examples to review this classic view.

Example: 1

Consider the equation

$$y_t = y_{xx} - y_x$$

With the initial condition

$$y(x,0) = e^{-x} + x,$$

The boundary conditions

$$y(0,t) = 1, \text{ and}$$

$$y_x(0,t) = e^{-1} - 1.$$

We solve this equation Homotopy Perturbation Method using different L and v_0.

Our choice is $L\phi = \frac{2}{x}\frac{\partial \phi}{\partial x}$ and $v_0 = 0$.

So we have the following equation

$$\phi_t + q\{-\phi_{xx} + \phi_x\} = 0.$$

Then equating the terms with identical powers of q, and then solving the resulting system of equations one has

$$u_0(x,t) = e^{-x} + x,$$

$$u_1(x,t) = 2e^{-x}t + t,$$

$$u_2(x,t) = 4\frac{t^2}{2!}e^{-x},$$

$$u_3(x,t) = 8\frac{t^3}{3!}e^{-x},$$

$$\vdots$$

So we have the approximation solution

$$y(x,t) = e^{-x} + x\left(1 + t + 4\frac{t^2}{2!} + 8\frac{t^3}{3!} + \cdots\right),$$

Yielding the closed form

$$y(x,t) = e^{-x} + xe^t,$$

Which is the exact solution.

Example: 2

Consider the equation $y_t + y_{xx} = y_{xxxx}$

With the initial condition

$y(x,0) = \sin(x)$,

We have the following homotopy equation

$\phi_t + q\{\phi_{xx} - \phi_{xxxx}\} = 0.$

Then equating the terms with identical powers of q, and then solving the resulting system of equations one can see

$u_0(x,t) = \sin(x)$,

$u_1(x,t) = 2\sin(x)\,t$,

$u_2(x,t) = 2t^2 \sin(x)$,

$u_3(x,t) = 4\dfrac{t^3}{3!}\sin(x)$,

$\vdots$

So we have the approximation solution as follows

$y(x,t) = \sin(x)\left(1 + 2t + 2t^2 + 4\dfrac{t^3}{3!} + \cdots\right) = \sin(x)\,(e^{-t}).$

Which is the exact solution.

Example: 3

Consider the equation $y_{tt} - y_{xx} = y_{xxxx} + y_x$

$y(x,0) = 1 + \sin(x)$, and $y_t(x,0) = 0.$

Our choice leads to the following homotopy equation

$\phi_{tt} + q\{-\phi_{xx} - \phi_{xxxx} - \phi_x\} = 0.$

Then equating the terms with identical powers of q, and then solving the resulting system of equations one can see

$u_0(x,t) = 1 + \sin(x)$,

$u_1(x,t) = \dfrac{t^2}{2!}\cos(x)$,

$u_2(x,t) = -\dfrac{t^4}{4!}\sin(x)$,

$u_3(x,t) = -\dfrac{t^6}{6!}\sin(x)$,

$\vdots$

So according to (8) approximate solution is

$y(x,t) = \sin(x) + \cos(x)\left\{1 + \dfrac{t^2}{2!} + \dfrac{t^4}{4!} + \dfrac{t^6}{6!} + \cdots\right\}$,

Which yields the closed form $y(x,t) = \sin(x) + \cos(x)(e^{-t})$. so we have the same terms as obtained.

Example: 4

Consider the equation $y_t + y_x = y_{xx} - y$

With the initial condition

$y(x,0) = e^{-x}$,

We have the following homotopy equation

$\phi_t + q\{\phi_x - \phi_{xx} - \phi\} = 0.$

Then equating the terms with identical powers of q, and then solving the resulting system of equations one has

$u_0(x,t) = te^{-x}$,

$u_1(x,t) = \left(\dfrac{t^2}{2!}\right)e^{-x}$,

$u_2(x,t) = \left(\dfrac{t^3}{3!}\right)e^{-x}$,

$u_3(x,t) = \left(\dfrac{t^4}{4!}\right)e^{-x}$,

$\vdots$

So we have the approximation solution

$y(x,t) = \left(1 - t + \dfrac{t^2}{2!} - \dfrac{t^3}{3!} + \dfrac{t^4}{4!} + \cdots\right)e^{-x}.$

Which is the exact solution.

Example: 5

Consider the equation

$y_t = y_{xx} + y_x$

With the initial condition

$y(x,0) = e^x.$

We have the following homotopy equation

$\phi_t + q\{-\phi_{xx} - \phi_x\} = 0.$

Then equating the terms with identical powers of q, and then solving the resulting system of equations one can see

$u_0(x,t) = e^x,$
$u_1(x,t) = 2te^x,$
$u_2(x,t) = 2t^2 e^x,$
$u_3(x,t) = 4\frac{t^3}{3!} e^x$
$\vdots$

So we have the approximation solution as follows

$$y(x,t) = e^x \left(1 + 2t + 2t^2 + 4\frac{t^3}{3!} + \cdots\right),$$

Yielding the closed form $y(x,t) = e^{x+t}$, which is the exact solution as obtained by homotopy perturbation method.

5. Conclusions

[1] In this paper, we proposed some guidelines for beginners who intend to solve their problems using the homotopy perturbation method.

[2] In the sequel we comparatively reviewed procedures which are used by researchers, through two examples.

[3] Then we presented a simple way to choose L and v_0 when we use the homotopy perturbation method to solve time-dependent differential equations.

[4] In most cases, our simple choice yields exact an solution or at least very good approximations.

[5] Although there are examples that show our choice isn't as good as other choices, it still produces convergent series that makes it a reliable one in solving a wide class of functional equations.

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