IMPLEMENTATION OF MODIFIED SIMULTANEOUS ALGEBRAIC RECONSTRUCTION TECHNIQUE

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Abstract : ***In this paper we introduced the Modified Simultaneous Algebraic Reconstruction Technique (MSART), in this we tried to modify the reconstruction in single iteration with minimizing the noise and also simultaneously tried to accelerate the algorithm in the sense that it reaches faster to the solution. Heuristically it is shown in this paper it will converge to actual solution and since in a single iteration we are combining M (number of rays in projection data) iteration of simple ART, which saves many calculations required, if those iterations were done separately, thus convergence is faster. In case of simple ART in every iteration, calculation of projection data requires larger storage space with extra calculations, while in this modified method, where we are combining the iterations, the extra large storage space is not required. Moreover we are modifying the projection data for each ray simultaneously, thus making the convergence faster with no storage space required for storing projection data in every iteration. But, still we have shown these properties heuristically, and nor proved them mathematically, specially the convergence. Thus in this paper , we will implement the algorithm on some test images with calculated projection data. The convergence will be shown with respect to the errors based on discrepancy measures defined in [3,4,6] .***
First we will give test patterns and their projection data, next our algorithm and the errors considered for stopping rule and for convergence will be given. In last section the test results and output will be shown.

Key words: Reconstruction techniques, Algebraic Reconstruction Method, Convolution Back Projection Method, MSART , Tomography, Image processing

I. INTRODUCTION

In this paper we introduced the Modified Simultaneous Algebraic Reconstruction Technique (MSART), which was implemented on certain test images in last section . The quality of reconstruction and the speed of convergence has been shown to improve and suitability of proposed method and its fast convergence has been shown with geometrical method and proved heuristically as well. In Section 4 the algorithm proposed in section 3 has been implemented computationally on some test images with projection data calculated numerically with the help of computers. In this implementation the noise factor other than numerical approximation has not been taken. In present chapter we will compare our results with that of other algorithms. For comparison we chose the Convolution Back projection (CBP). The CBP algorithm is probably the most widely used of all methods, as it is claimed to be fast, efficient, reasonably accurate, and easy to implement on computer. The next section provides the comparison with respect to both in terms of error analysis and digital quality of reconstruction as well.

COMARISON WITH CBP

For the purpose of comparison again we have the same three test images of section 5.2.1, The first test image named as PIC1 is face of a girl, Srivastava (1994) second test image named here as PIC2 is cross section of Brain, and PIC5 are Shepp Phantoms, Shepp and Logan (1974). The first four images are 64X64 digitized models and the fifth is 128X128 digitized model. All these test images were shown in Figure 5.2.1 and their digitized projection data is shown in Figure 5.2.2. [11,13]

MODIFIED SIMULTANEOUS ALGEBRAIC TECHNIQUE (MSART)

This method is modification of projection method. We had already seen some modification as relaxation method SIRT or SART described in Chapter-3. Our modification is to improve convergence rate to reach to solution inspite of noise present in the projection data. This method is combination of SART with other modifications. In ART given [16,3,6]

$$f_j^{(k+1)} = f_j^{(k)} + a_j \frac{p_i - \sum_{j=1}^{n} a_{ij} f_j^k}{\sum_{j=1}^{n} a_{ij}^{-2}}$$

$$\forall j \notin \quad \mathrm{a}_{ij} \neq D$$

$$\mathrm{i} = \mathrm{k} \ (\mathrm{mod}\ \mathrm{M}) + 1$$

for solving the system of linear algebraic equations.

$$\underline{A}\underline{f} = \underline{p} \qquad \text{with A} = (a_{ij})_{MXN}, \underline{f} = (f_j)_{j=1}^{N}$$

$$\underline{p} = (p_i)_{i=1}^{M}$$

in each iteration only those fj's are modified which intersect with ith ray and in one iteration only one ray is considered. Thus it makes to do a large number of iterations for getting some significant results and making the rate of convergence very slow. Also it gives noise as ($\forall$).

To improve upon these difficulties simultaneous algebraic reconstruction technique (SART) was introduced by Anderson (1984) [measurements are computed by noise and give inconsistency in the system as well].

In this method to reduce noise, the f_j's are not modified in each ray at one iteration but the modification only is stored separately for all rays of projection measurements and then in one iteration the modification of all rays for j^{th} value is been incorporated at the end. Thus get the formula as

$$f_j^{(kH)} = f_j^{(k)} + \frac{\sum_{i=1}^{M} \left[a_{ij} \frac{p_i - \sum_{j=1}^{N} a_{ij} f_j^{(k)}}{\sum_{i=1}^{M} a_{ij}^2} \right]}{\sum_{i=1}^{M} a_{ij}} \forall_j = 1, ---N$$

This formulation actually makes one iteration more time and cost consuming, but reduces noises. Thus in applications where accurate estimates are required on the cost of expensive iteration, this is suitable, its convergence was also proved by Ming (2005). Now we introduce our method.

The basic principle behind this is first the reconstruction space is taken to be square, thus if it is of any shape we cover it by square grid and pixels were also taken to be square and measurements are reconstruction region. The strips are first approximated as lines i.e. the strip width is neglected negligible.

Unlike ART where measurements pi are numbered in a manner and then iterations were moving in cyclic manner over these numbered rays, in this method we do not require to number the measurements in any particular manner. Here we all require to have aij's are guding force for modification of image values, we first start with each ray and modify all fj's which is intersecting that ray, when we move to next ray we modify all pixel elements of this ray, which may results some of fi's are again modified. The modification which we are introducing at each ray is similar to constrained ART, at first we introduce only partial constraint that is all image values and measurements are non negative. The modification in each ray is given for all fi's which intersect that ray so a typical ith ray modification is

$$\frac{p_i - \sum_{j=1}^{N} a_{ij} f_j^{*(k)}}{\sum_{j=1}^{N} a_{ij}^2} a_{ij}$$

which is added to all fj's such that aij # 0 here $f_j^{*(k)}$ is used instead of $f_j^{(k)}$ which is either $f_j^{(k)}$ if fj was not modified in any of previous rays or it is modified. $f_j^{(k)}$ if it has been modified previously. Then the elements fj of ith ray are modified as

$$\max \left\{ 0,\ \mathrm{f}_j^{*(k)} + r_{ij} \frac{p_i - \sum_{j=1}^{N} a_{ij} f_j^{*(k)}}{\sum_{j=1}^{N} a_{ij}^2} \right\}$$

This modification keeps for each ray the non-negativity constraint.

Once this process is completed for all rays i = 1, ---- M, the final out come $f_j^{(k+1)}$, j = 1, ---N is said to be (k+1)st estimate obtained for kth estimate.

This modification actually makes the convergence process fast and computationally also requires less storage capacity. Next we will Test some images .

1 TEST OBJECTS

1.1 TEST IMAGES

For the purpose of numerical implementation of the introduced new algorithm MSART, we took three test images. The first test image named as PIC1 is face of a girl, Srivastava (1994) [23,24] second test image named here as PIC2 is cross section of Brain, and PIC3 are Shepp Phantoms, Shepp and Logan(1974) [9]. The first two images are 64X64 digitized models and the third is 128X128 digitized model. All these test images are shown in Figure 1.1.

PIC1: Face of Gir PIC2: Cross Section of Brain PIC5: Shepp Phantom

Figure 1.1: Digitized test images first two are 64X64 digitized images and last is 128X128 digitized image

2 PROJECTION DATA

The projection data for these test images is calculated as line integral on unit circle and for each line (s, θ), the line integral given as this integral is numerically calculated with using Simpson's method. For images PIC1 to PIC2 the size of projection data is 100X64 that is 100 views (θ) and 64 lines (s) for each view. But for PIC3 the projection data size is taken as 200X128, 200 views (θ) and 128 lines (s) for each view. The digitized images of the calculated projection data are shown in Figure 2.1.

PIC1 PIC2 PIC3

Figure 2.1: 100X64 digitized images of projection data Digitized projection data for test images PIC1-PIC2 and 200X128 digitized projection data for PIC3

3 ERRORS

The error considered here for testing the convergence to actual solution were taken for both the discrepancy measures between projections and successive reconstructions of the test images which are:

Discrepancy Measure between the Projections

$$EP1 = \sum_{i=1}^{M} \left| p_i - p_i^k \right|$$

$$EP2 = \left(\sum_{i=1}^{M} \left(p_i - p_i^k \right)^2 \right)^{1/2}$$

where, p_i^k = i^{th} ray sum of k^{th} iteration estimate

$$\Rightarrow p_i^k = \sum_{j=1}^{N} a_{ij} f_j^{(k)},\ i=1,2,\ldots,M$$

and p_i = the original i^{th} ray projection data

Discrepancy in Consecutive Estimates of image

$$EF1 = \sum_{j=1}^{N} \left| f_j^{(k+1)} - f_j^{(k)} \right|$$

$$EF2 = \left[\sum_{j=1}^{N} \left(f_j^{(k+1)} - f_j^{(k)} \right)^2 \right]^{1/2}$$

4 MSART ALGORITHM

In this paper MSART is introduced here we provide the scheme of the algorithm:

Algorithm 4.1

1. Begin
2. Get the projection data with M
3. Get stopping criterion as e
4. Start with iteration (k)=0
5. Get Initial Guess with c as cons
   $\mathbf{f}^0 = (c, \ldots\ldots\ldots, c)'$,
6. i=0
7. $f_j^{1(k+1)} = f_j^{*(k)} + a_{ij} \dfrac{p_i - \sum_{j=1}^{N} a_{ij} f_j^{*(k)}}{\sum_{j=1}^{N} a_{ij}^2}$
8. $f_j^{(k+1)} = \max\left\{0,\ f_j^{1(k+1)}\right\}, \quad \forall j \ni a_{ij} \neq 0$
9. i=i+1, check (i<M) if yes repeat step 7, 8
10. Else k=K+1,
11. Calculate $EP2 = \left(\sum_{i=1}^{M} \left(p_i - p_i^k \right)^2 \right)^{1/2}$ or
    $EF2 = \left[\sum_{j=1}^{N} \left(f_j^{(k)} - f_j^{*} \right)^2 \right]^{1/2}$
12. check (EP2>e) or (EF2>e) If yes repeat steps 6 to 11
13. Else Stop
14. END

5 RESULTS OF NUMERICAL IMPLEMENTATION

In this section implementation of the above defined MSART algorithm 4.1 and 4.2 is being done on test images of in section 3.1 with projection data given in section 5.2.2. The reconstruction, convergence using the discrepancy measures between projections

EP1 and EP2 and the discrepancy measures between successive reconstructions of test images are provided in tables. The detail of output is:

PIC1: Reconstruction results at different iterations in figure 5.1.1 and convergence tables for the discrepancy measures between projections EP1 and EP2 are provided in tables 5.1.2, and for the discrepancy measures between successive reconstruction of test images EF1 and EF2 are given in table 5.1.3.

PIC2: Reconstruction results at different iterations in figure 5.2.1 and convergence tables for the discrepancy measures between projections EP1 and EP2 are provided in tables 5.2.2, and for the discrepancy measures between successive reconstruction of test images EF1 and EF2 are given in table 5.2.3.
PIC3: Reconstruction results at different iterations in figure 5.3.1 and convergence tables for the discrepancy measures between projections EP1 and EP2 are provided in tables 5.3.2, and for the discrepancy measures between successive reconstruction of test images EF1 and EF2 are given in table 5.3.3.

PIC1 :

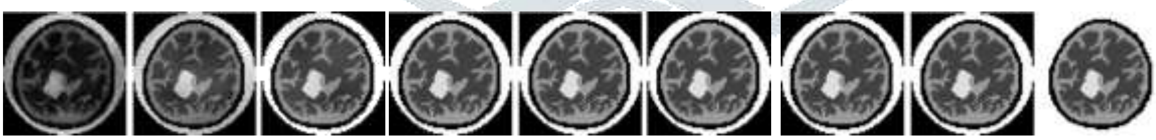
ITERATION 1 ITERATION 5 ITERATION 10 ITERATION 20 ITERATION 30 ITERATION 40 ITERATION 50 ITERATION 60 Original

Figure 5.1.1: Reconstruction of PIC1 at different iterations

ITERATIO NS	EP1	EP2	ITERATION	EF1	EF2
0	519.9	687.981	0	71.6932	114.156
1	397.53	512.023	1	28.829	45.667
2	285.658	378.796	9	3.59461	6.30248
3	243.278	309.185	10	2.84848	4.94335
10	39.6143	52.5184	25	0.0304446	0.0520588
20	2.10063	2.73198	30	0.00776078	0.0139992
30	0.22885	0.302257	40	0.00104117	0.0024915
49	0.0360264	0.0468442	41	0.000916861	0.0022222
50	0.0332426	0.0432643	58	0.000170983	0.000500484
59	0.0168351	0.0219124	59	0.000157819	0.000463709

Table 5.1.2: convergence with projection data PIC1 **Table 5.1.3**: convergence in consecutive estimates PIC1

PIC2:



ITERATION 1 ITERATION 5 ITERATION 10 ITERATION 20 ITERATION 30 ITERATION 40 ITERATION 50 ITERATION 60 Original

Figure 5.2.1: Reconstruction of PIC2 at different iterations

ITERATION	EP1	EP2	ITERATION	EF1	EF2
0	74.7791	101.445	0	9.77688	15.8067
1	58.6512	77.5108	1	5.03446	8.01841
10	10.413	13.317	10	0.705458	1.04219
20	1.41774	1.79767	20	1.41774	1.79767
21	1.17079	1.49238	21	1.17079	1.49238
30	0.246553	0.329573	30	0.246553	0.329573
49	0.0188832	0.0251376	49	0.0188832	0.0251376
50	0.0167743	0.0226385	50	0.0167743	0.0226385
58	0.00822146	0.0112213	58	0.00822146	0.0112213
59	0.00770095	0.0104277	59	0.00770095	0.0104277

Table 5.2.2: convergence with projection data PIC2 **Table 5.2.3**: convergence in consecutive estimates PIC2

PIC3:

ITERATION 10 ITERATION 20 ITERATION 30 ITERATION 40 ITERATION 50 ITERATION 60 Original

Figure 5.3.1: Reconstruction of PIC3 at different iterations

ITERATION	EP1	EP2	ITERATION	EF1	EF2
0	547.614	756.139	0	29.5712	60.5521
1	254.696	378.413	1	13.0262	25.2493
10	11.892	17.3521	10	0.507816	0.953837
20	1.0725	1.50108	20	0.037813	0.0729004
30	0.114655	0.155026	30	0.00341378	0.00648435
40	0.0258352	0.0357468	40	0.00042556	0.000895449
49	0.0106065	0.0144136	49	9.11316e-005	0.000249154
50	0.00924201	0.0127918	50	7.89933e-005	0.000222253
58	0.00439942	0.00609243	58	3.11008e-005	9.74378e-005
59	0.00407651	0.00560321	59	2.85391e-005	8.90986e-005

Table 5.3.2: convergence with projection data PIC3 Table 5.3.3: convergence in consecutive estimates PIC

6 DISCUSSION AND CONCLUSIONS

6.1 ERROR ANALYSIS

The error considered for error analysis, are L1 and L2 errors in reconstruction at different iterations and original test image. They are represented as EL1 and EL2 in result tables.

TABLE 6.1: L_1 and L_2 errors for PIC1

ITERATION	EL1	EL2
1	20.7321	33.5953
2	14.1084	20.9319
3	9.9297	15.7198
5	6.08287	9.54181
8	3.02308	4.92604
10	1.9299	3.18229
15	0.556968	0.944499
20	0.174735	0.359911
25	0.0696505	0.237065
30	0.0364597	0.208664
40	0.0163477	0.197379
50	0.0105032	0.195736
60	0.0081766	0.195409
CBP	7.91698	13.4205

TABLE 6.2: L_1 and L_2 errors for PIC2

Iteration	EL1	EL2
1	3.34553	5.19102
3	1.78953	2.79446
5	1.17058	1.7198
10	0.445308	0.651783
15	0.186997	0.291063
20	0.0812621	0.148123
30	0.0230243	0.0915325
40	0.0113048	0.0844714
50	0.00701021	0.0828324
60	0.00497559	0.0823621
CBP	2.08906	3.34038

TABLE 6.3: L_1 and L_2 errors for PIC3

Iteration	EL1	EL2
1	13.3526	23.8421
3	4.78426	8.56789
5	2.32305	4.10408
10	0.50985	0.89246
15	0.151073	0.273642
20	0.0516122	0.10114
30	0.00870633	0.0271686
40	0.00239777	0.0189209
50	0.000959122	0.0179874
60	0.000527374	0.0178639
CBP	6.66922	14.161

6.2 PICORIAL QUALITY OF RECONSTRUCTION

In this section the digitized images of reconstruction at various iteration with reconstruction by convolution back projection algorithm are compared the results are shown in figures 6.1 to 6.5 for five test images PIC1 to PIC3 respectively.

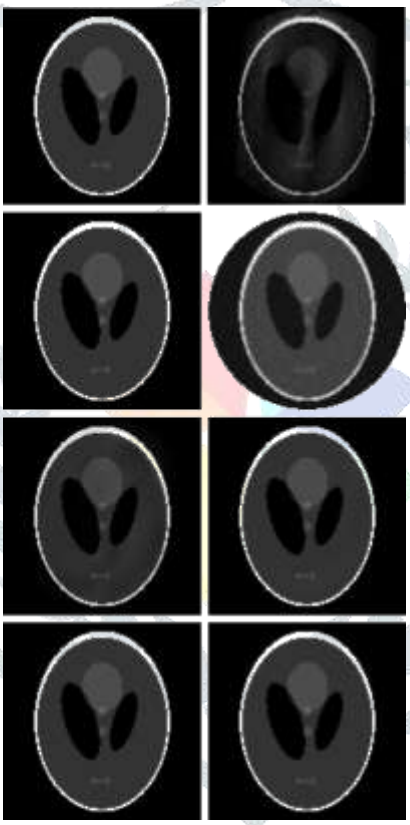


Figure 1: Comparison of reconstruction with CBP for PIC1 From Left to Right: Row1--Original image, Reconstruction by SMART in iterations 1, 5, 10, 20, Row2-- iterations 60, reconstruction by CBP, iterations 30, 40 50.

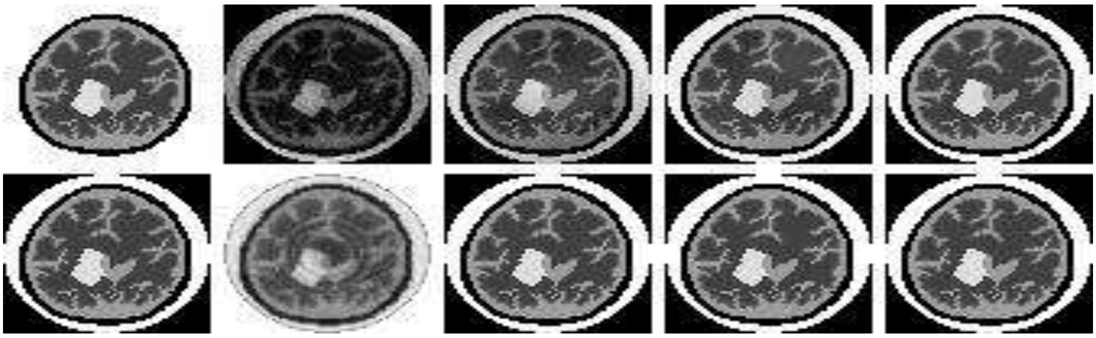


Figure 2: Comparison of reconstruction with CBP for PIC2 From Left to Right: Row1--Original image, Reconstruction by SMART in iterations 1, 5, 10, 20, Row2-- iterations 60, reconstruction by CBP, iterations 30, 40 50.

Figure 3: Comparison of reconstruction with CBP for PIC5 From Left to Right: Row1--Original image, Reconstruction by SMART in iterations 1, Row2-- iterations 60, reconstruction by CBP, Row3-- iterations 5, 10, Row4-- iterations 50, 40.

CONCLUSION

From the discussion in last section it is evident that the proposed algorithm improves the quality of reconstruction upon convolution back projection algorithm both visually and in terms of error analysis. From tables 6.1 to 6.3 it is observed that in terms of error analysis the quality of reconstruction is better as early as from 4th or 5th onwards. In pictorial view also the reconstruction by MSART is better than CBP.

Further this algorithm can be modifies for use in limited view projection data problems as well, where it looks to work better that transform methods.

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