REPRESENTATION OF RISK MEASURES AS CHOQUET INTEGRALS

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Abstract : In this paper we present necessary and sufficient conditions for the representability of a *coherent risk measure* ρ on $\mathbf{\Omega}$ (Ω, F, P) as the Choquet integral. The main objective is to study coherent risk measures which satisfy additionally the axioms of dilatation monotone and comonotonic additivity.

Keywords: Choquet integral, comonotonic additive, comonotone, Fatou property, coherent risk measure, dilatation monotone.

0. Introduction

The notion of measure is a very important concept in mathematics, particularly for the theory of integrals. These measures are based on the property of additivity. This property has been abandoned in many areas such as in decision theory and in the theory of cooperative games. It becomes essential to define nonadditive measures, which are usually called capacities [2] or fuzzy measures [13]. A fundamental concept that uses these nonadditive measures is the Choquet integral [2], defined as an integral with respect to a capacity. The Choquet integral is known as a nonadditive integral of a function with respect to a capacity (or nonadditive measure, or fuzzy measure). It was characterized mathematically by Schmeidler [11], and then by Murofushi and Sugeno [10] using the concept of the capacity introduced by Choquet. Later it was used in utility theory [12], leading to the so-called Choquet Expected Utility.

So far, many studies have focused on the theory and the applications of the Choquet integral defined on a discrete set (Faigle and Grabisch [6], Grabisch and Labreuche [7]). In the discrete case, the Choquet integral of a function with respect to a capacity is easy to calculate. However, this is not the case for the Choquet integrals of functions on a continuous support. Recent developments that have been conducted on the Choquet integral of real functions [14, 15] appear to open up new horizons.

In this paper our purpose is to present necessary and sufficient conditions for the representability of a coherent risk measure. The main objective is to study coherent risk measures which satisfy additionally the axioms of dilatation monotone and comonotonic additivity.

1. Notations

We work with a probability space $\mathbf{\Omega} = (\Omega, F, P)$. Denote by $M = M(\mathbf{\Omega})$ the set of probability measures on (Ω, F) which are absolutely continuous w.r.t. P. We endow with the topology inherited from L^1 via the inclusion $Q \to \frac{dQ}{dP}$, $Q \in$ M. Denote by $\sigma(X)$ for $X \in L^0$ the σ-algebra generated by the sets $\{X > c\}$ for all $c \in R$ and all the sets of null probability.

2. Coherent Risk Measures

The important notion of a coherent risk measure was introduced by Artzner et al. (1999) [1].

Definition 1. A functional $\rho : L^\infty \to$ R is called a coherent risk measure if it satisfies the following properties:

1. Translation Invariance: $\rho(a + X) = \rho(X) - a$ for all $X \in L^\infty$ and $a \in$ R.
2. Positive Homogeneity: $\rho(aX) = a\rho(X)$ for all $X \in L^\infty$ and $a \geq 0$.
3. Monotonicity: For $X \in L^\infty$ with $X \geq 0$ almost surely, $\rho(X) \leq 0$ holds.
4. Sub-Additivity: $\rho(X + Y) \leq \rho(X) + \rho(Y)$ for all $X, Y \in L^\infty$.

Coherent risk measures can be characterized as follows, see [4]:

Theorem 2. Let ρ be a coherent risk measure. The following properties are equivalent:

1. There exists a non-empty closed convex set $Q_\rho \subseteq$ M such that for all $X \in L^\infty$,
$\rho(X) = -\inf_{Q \in Q_\rho} EQ[X]$. (1.1)

2. $A_\rho : \{X \in L^\infty | \rho(X) \leq 0\}$ is closed in the weak* topology $\sigma(L^\infty, L^1)$.

3. ρ satisfies the Fatou property: For any sequence $(X_n)_{n\geq 1}$ of random variables, uniformly bounded and converging in probability to X, $\rho(X) = \liminf_{n\to\infty} \rho(X_n)$ holds.

4. For any sequence $(X_n)_{n\geq 1}$ of random variables, uniformly bounded and decreasing to X, $\rho(X) = \lim_{n\to\infty} \rho(X_n)$ holds.

For a non-empty closed convex set $Q \subseteq M$, the infimum in (1) can be replaced by a minimum iff one of the following conditions holds

5. Q_ρ is weakly compact in L^1.

6. If $(A_n)_{n\geq 1}$ is a decreasing sequence of sets $A_n \in$ F such that $\bigcap_{n\geq 1} A_n = \phi$, then $\lim_{n\to\infty} \sup_{Q\in Q_P} E_Q[1_{A_n}] = 0$.

7. If $(A_n)_{n\geq 1}$ is an increasing sequence of sets $A_n \in$ F such that $\bigcap_{n\geq 1} A_n = \Omega$, then $\lim_{n\to\infty} \sup_{Q\in Q_P} E_Q[1_{A_n}] = 1$.

8. ρ satisfies the *Lebesgue property*, i.e. for any uniformly bounded sequence $(X_n)_{n\geq 1}$ of random variables, converging in probability to X, $\rho(X) = \lim_{n\to\infty} \rho(X_n)$.

Furthermore, the mapping $\rho \to Q_\rho$ from the set of coherent risk measures with the Fatou property into the set of all non-empty closed convex subsets of M is one-to-one and onto.

Remark 3. Note that coherent risk measures defined on L^p for $1 \leq p < \infty$ are easily characterized using the Hahn–Banach Representation Theorem. The point in the above result is that in (1.1) the infimum is taken over a subset of M, i.e. σ-additive probability measures, and not over a set of *finitely additive* positive measures with total mass 1 in the dual of L^∞.

3. Dilatation monotone and comonotonic additive risk measures

Definition 4. A coherent risk measure ρ is called *dilatation monotone* if for all $X \in L^\infty$ and any σ-algebra $\tilde{F} \subseteq F, \rho(X) \geq \rho(E[X \mid \tilde{F}])$ holds.

This definition captures the economic intuition that averaging out a risky return should never increase the involved risk, see [9].

Definition 5. (See [8, 11, 16]) A pair of random variables $X, Y \in L^0$ is said to be *comonotone* if

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega') \geq Q.$$

A functional (in particular a risk measure) $F : L^\infty \to$ R is called *comonotonic additive*, if $F(X + Y) = F(X) + F(Y)$ holds for all comonotonic pairs $X, Y \in L^\infty$.

The assumption of comonotonic additivity for risk measure appears to be economically natural since it implies that we cannot diversify risk by combining comonotone claims.

We can characterize the coherent dilatation monotone risk measures with the Fatou property in terms of the sets Q_ρ and A_ρ from Theorem 2 as follows:

Theorem 6. Let ρ *be* a coherent risk measure, satisfying the Fatou property. The following properties are equivalent.

1. ρ is dilatation monotone.
2. $X \in A_\rho$ implies $E[X \mid \tilde{F}] \in A_\rho$ for all σ-algebras $\tilde{F} \subseteq F$.
3. For all $Q \in Q_\rho$ and any σ-algebra $\tilde{F} \subseteq F$ there exists $Q \in Q_\rho$ such that $E\left[\frac{dQ}{dP} \mid \tilde{F}\right]$.

Proof: Assume ρ to be dilatation monotone. For $X \in A_\rho$ and a σ-algebra $\tilde{F} \subseteq F$ we have $\rho(E[X \mid \tilde{F}]) \leq \rho(X) \leq 0,$ hence $E[X \mid \tilde{F}] \in A_\rho,$ i.e. condition (2) holds.

Let property 2 hold. For all $X \in L^\infty$ we have $\rho(X + \rho(X)) = 0$, hence $X + \rho(X) \in A_\rho$. Therefore, by assumption $E[X + \rho(X) \mid \tilde{F}] \in A_\rho$ and thus $\rho(E[X \mid \tilde{F}]) \leq \rho(X)$.

To check that property 2 implies 3 set

$$A_\rho^0 = \left\{Y \in L^1 \mid \forall X \in A_\rho : E[XY] \geq 0\right\}.$$

Note that $E\left[E[Y \mid \tilde{F}]X\right] = E\left[YE[X \mid \tilde{F}]\right] \geq 0$ by assumption. Hence $Y \in A^\circ_\rho$ implies $E\left[Y \mid \tilde{F}\right] \in A_\rho^0$. By the bipolar theorem, see Delbaen ([4], Chapter 4.1), we have $\left\{\frac{dQ}{dP} \mid Q \in Q_\rho\right\} = A_\rho^0 \cap \left\{Y \in L^1 \mid E[Y] = 1\right\},$ and property 3 follows.

Similarly, property 3 implies 2 since $A_\rho^0 = \left\{\lambda \frac{dQ}{dP} \mid Q \in Q_\rho, \lambda \geq 0\right\}$.

Coherent comonotonic additive risk measures with the Fatou property can be characterized in terms of the set Q_ρ as follows:

Theorem 7. If ρ is a coherent risk measure with the Fatou property, then the following properties are equivalent:

1. ρ is comonotonic additive.
2. For any two events $B_1 \subset B_2 \subset \Omega$ there exists $Q^* \in Q_\rho$ such that $Q^*(B_i) = \sup_{Q \in Q_\rho} Q(B_i)$ for $i = 1, 2$.
3. For any finite family of events $(B_i)_{i \in I}$ such that either $B_{i1} \subset B_{i2}$ or $B_{i2} \subset B_{i1}$ for all $i_1, i_2 \in I$ there exists $Q^* \in Q_\rho$ such that $Q^*(B_i) = \sup_{Q \in Q_\rho} Q(B_i)$ for all $i \in I$.

Moreover, if ρ satisfies the Lebesgue property, then the family of events $(B_i)_{i \in I}$ in property 3 can be infinite.

We shall prove this theorem using Theorem 11 from Delbaen [4], arguments of Lemma 2 from Delbaen [4] and some results about the Choquet integrals.

Proof. Let us prove that property 2 implies 1. That

$$\rho(1_{G \cup H}) + \rho(1_{G \cap H}) \leq \rho(1_G) + \rho(1_H) \text{ for all } G, H \in F.$$

Set $B_1 = \Omega \setminus (G \cup H)$ and $B_2 = \Omega \setminus (G \cap H)$. Since $B_1 \subset B_2$ by assumption there exists $Q^* \in Q_\rho$ such that $Q^*(B_i) = \sup_{Q \in Q_\rho} Q(B_i)$ for $i = 1, 2$. Hence

$$Q^*(G \cup H) = \inf_{Q \in Q_\rho} Q(G \cup H) \text{ and } Q^*(G \cup H) = \inf_{Q \in Q_\rho} Q(G \cap H).$$

So using (1) we get

$$\rho(1_{G \cup H}) + \rho(1_{G \cap H}) = -Q^*(G \cup H) - Q^*(G \cap H)$$
$$= -Q^*(G) - Q^*(H) \leq \rho(1_G) + \rho(1_H).$$

So we proved that properties 2 implies 1. Clearly property 3 implies 2, so it remains to check that 1 implies 3.

Let ρ be comonotonic additive. Clearly, the indicator functions of B_i, $i \in I$ are comonotone, so we have

$$\rho\left(-\sum_{i \in I} 1B_i\right) = \sum_{i \in I} \rho(-1B_i).$$

By Theorem 2 and Theorem 11 from Delbaen [4] there exists $Q^* \in Q_\rho$ such that $\rho\left(-\sum_{i \in I} 1_{B_i}\right) = -E_{Q^*}\left[-\sum_{i \in I} 1_{B_i}\right]$ and thus $\sum_{i \in I} \rho\left(-1_{B_i}\right) = \sum_{i \in I} E_{Q^*}\left[-1_{B_i}\right]$. But $\rho(-1_{B_i}) = -\inf_{Q \in Q_\rho} E_Q[-1_{B_i}] \geq E_{Q^*}[1_{B_i}]$ for each $i \in I$ and hence $\rho(-1_{B_i}) = E_{Q^*}[1_{B_i}] = Q^*(B_i)$ for each $i \in I$. So $Q^*(B_i) = \rho(-1_{B_i}) = \sup_{Q \in Q_\rho} Q(B_i)$ for all $i \in$ I.

Now assume that ρ satisfies the Lebesgue property and I may be infinite. For a finite subset $J \in I$ set

$$Q_\rho(J) = \left\{ Q \in Q_\rho : Q^*(B_i) = \sup_{Q \in Q_\rho} Q(B_i) \forall_i \in J \right\}.$$

We have proved that the set $Q_\rho(J)$ is non-empty whenever J is a finite subset of I. Clearly $Q_\rho(J)$ is also a (weakly) compact subset of Q_ρ. Thus since by Theorem 2 the set Q_ρ is compact and all finite intersections of the sets $Q_\rho(J_\alpha)$ are non-empty, whenever J_α are finite, it follows that there exists a probability measure

$$Q^* \in \bigcap_{\text{all finite subsets } J \text{ of } I} Q(J) \neq \phi.$$

Clearly Q^* satisfies $Q^*(B_i) = \sup_{Q \in Q_\rho} Q(B_i)$ for all $i \in I$.

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