

Numerical Illustration of Sensitivity and Parametric Analysis of Linear Programming Model

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Abstract: *The inevitable task in the development of realistic linear programming model is the collection of accurate and reliable numerical values for the coefficients. In any occasion, only estimated or average values of the coefficients are available. Therefore, we are compelled to estimate the problem to be determined. Hence, it is important to study the behaviour of solutions to a linear programming problem when the coefficients of the values are allowed to fluctuate. To find the optimality and to determine what happens to the optimal solution when certain changes are made in the system is crucial. We are still able to observe what happens to the linear programming coefficients under investigation as they continuously change their values that are linear functions of a parameter. Moreover, we would like to determine the effects of those changes without having to solve a new problem or a series of new problems afresh with respect to the established lemmas.*

Keyword: *Linear Programming LP, Optimality, Coefficients, Iteration and parameter.*

Introduction

A Mathematical Model Comprises of independent and dependents variables and a system of relationship in the form of equations or inequalities that exist between the variables. Mathematically, the numerical methods employed to solve the equations underlying the mathematical model are often the important aspect of the model development process. Mathematical models include variable parameters which constitute the relationship that unfold from experiments and as such, their actual values are not known precisely but may vary within some ranges of uncertainty. The parametric analysis of a mathematical model becomes even more complicated if the numerical calculations cannot be handled with known methods. The simplest and also the most common procedure for assessing the effects of parameter variations on a model's result is to vary selected input parameters, rerun code and record the corresponding changes in the result. The model parameters responsible for the largest relative changes are classified to be the most important. For complex models though, the large amount of computing time needed by such re-calculations severely restricts the scope of this analysis. In practice, this means that the modeler can investigate only a few parameters that he judges a priori to be important.

There are certain questions that are often asked regarding optimal solution of a linear programming (LP) problem. For example, what happens to the optimum solution (both the values of the variable and the value of the objective function) when certain changes occur in some of the values of the original data? These are some of the questions that parametric and sensitivity analysis tries to answer. If the objective function or the variables change when an original coefficient is changed, then we say that the optimal solution of the programming is sensitive.

In parametric analysis, we observe the changes that take place in the optimal solution as the original values expressed as a linear function of a parameter are continuously changed. The maximum value of the parameter is such that it will ensure the feasibility and optimality conditions are satisfied.

In sensitivity and parametric analysis, one needs to define four main types of objects: the work, result, configuration and program objects. The work object contains the sequential code for doing one run of the computation with a specific set of parameters. It takes a configuration object representing the parameters and returns a result object representing the output of the computation for that particular configuration. The program object is responsible for creating a list of parameter configurations that the programmer wants to try and then adding work objects with each of these configurations to the work pool so that they can be processed in parallel by the volunteer workers. After the parallel step, control returns to the program object, which then processes the results that have been collected and write them into a file, if desired. After processing these results, the programmer can proceed to try more configurations if desired, possibly depending on the results of the previous batch.

Discovering that the system behaviour greatly changes for a change in a parameter value can identify a leverage point in the model, a parameter whose specific value can significantly be the behaviour model of the system of management in an organization for the following reasons. It helps to determine which decision variables are critical. It is useful for control purposes and it helps to determine the range of values which certain decision variables can take on without affecting a change in the optimum solution (Egbon, 2009).

In linear programming the parameter (input data) of the model can change within certain limits without causing the optimum solution to change; in other words, parameters are usually not exact. With sensitivity analysis, we can ascertain the impact of this uncertainty on the quality of the optimum solution. The scope of linear programming does not end at finding the optimal solution to the linear model of a real life problem. Sensitivity analysis of linear programming continues with the optimal solution to provide additional practical insight of the model. Sensitivity analysis examines how sensitive the optimal solution is to changes in the coefficients of the linear programming (LP) model. This process is also known as post optimality analysis (Taha, 2006). Since we live in a dynamic world where changes occur constantly, this study of the effects on the solution due to changes in the data of a problem is very useful. The following are the types of changes in the data of a LP problem which can affect the feasibility/optimality of the present optimal solution.

- (i) Changes in the right-hand side (RHS) values of the linear constants
- (ii) Changes in the coefficients of the objective function.
- (iii) Changes in the technological coefficients of the decision variables.
- (iv) Addition of new variables to the problem.
- (v) Addition of new constants.

In general, these changes may result in one of the following outcomes:

- (i) The optimal solution remains unchanged that is, the basic variables and their values remain essentially unchanged.
- (ii) The basic variables remain the same but their values are changed.
- (iii) The basic solution changes completely.

The detection of these outcomes by sensitivity analysis is based mainly on the primal-dual properties (Forrester, 2001). For every maximization linear programming problem, there is a corresponding minimization linear programming problem and vice versa. This is because, for businessman to maximize profit is equivalent of his minimizing costs of operating the business. These pairs of closely related problems are called Dual linear programming problems. The first problem for which dual is sought is referred to as the PRIMAL problem, which is the LP model developed for a particular situation. The other related one is the dual one which is a closely related mathematical problem that can be derived directly from the primal problem (Ekoko, 1999).

When you use simplex method, you can obtain the optimal solution of the dual problem from the optimal tableau of the primal row. They are obtained along the z-row of the tableau and they correspond to the columns of the slack variables or the columns of the surplus variables (along the z-row). Furthermore, the optimum values of the objective functions of the primal and dual linear programming problems are equal. This fact can be used as a check whether you have arrived at optimal solution or not. Changes made in the original linear programming model will change the elements of the current optimal tableau, which in turn may affect the optimality or the feasibility of the current solution. This relationship will form the basis for economic interpretation of the linear programming model as well as for post-optimality analysis. The primal and dual solutions are so closely related that the optimal solution of either problem directly yields (with little additional computation) the optimal solution of the other. Thus, in an LP model in which the number of variables is considerably smaller than the number of constraints, computation savings may be realized by solving the dual from which the primal solution is automatically determined. The element of the row vector must appear in the same order in which the basic variables are listed in the Basic column of the simplex tableau (Agbadudu, 1996).

The feasibility of the current optimum solution may be affected only if the right-hand side of the constraints is changed and if a new constraint is added to the model. Infeasibility occurs when at least one element of the right-hand side of the optimal tableau becomes negative; that is, one or more of the current basic variables become negative (Taha, 2006). The optimality may be affected if there are changes in the original objective function coefficients and an addition of a new variable to the model. In some linear programming models, the values of the variables may be increased indefinitely without violating any of the constraints, meaning that the solution space is unbounded in at least one variable. As a result, the objective value may increase (maximization case) indefinitely. In this case, both the solution space and optimum objective value are unbounded. Unboundedness points to the possibility that the model is poorly constructed (Taha, 2006). In practical applications of linear programming (LP), the search for an optimal solution is not usually the only item of interest. We may not only be interested in determining the optimal solution but also to examine how the optimal solution behaves as changes are made in the system. In many practical (LP) problems, the importance of analysis of possible changes cannot be over emphasized because some of the data used in the model development are either uncertain or real. In the process of analysis, we may want to know how the uncertainty of the data influences the system and how changes in the real data affect the system. Analysis of discrete changes in model development using data to determine possible new optimum can be found (Ekoko, 2004).

A task in the development of realistic linear programming models is the collection of accurate and reliable numerical values for the coefficients. In some instances only estimates or average values of the coefficients are available. We are forced to use these point estimates and consider the problem to be deterministic. Hence, it is important to study the behaviour of solution to the linear programming problem when the coefficients of that variable are allowed to vary i.e. for what ranges of coefficient values will the deterministic solution remain optimal?

Most of the work done on variations in coefficients have been on sensitivity analysis, which focuses on what happens to the original optimal solution when a set of new values discretely replace the initial values. In parametric analysis we are able to observe what happens to the LP coefficient (under investigation) as they continuously change their values that are linear functions of a parameter (Ekoko, 2005).

In many practical problems, we want to find out not only an optimal solution but also to determine what happens to this optimal solution when certain changes are made in the system. We could like to determine the effects of those changes without having to solve a new problem. The changes to which the system is subjected in analyzing the incremental behaviour of the optimal solution can represent either real changes that can be made in the operation of the physical system which the linear programming model represents or fictitious changes which are made to investigate the effects of uncertainty in the basic data. The practical application of linear programming can be hampered at times by imperfect knowledge of the necessary data or by a complete lack of data (Ekoko, 2006).

Sensitivity and parametric analysis of LP problem can provide valuable management information to deal with uncertainties. The uncertainty ranges can be obtained by performing the different types of sensitivity analysis depending on the nature of the uncertainty: These include perturbation under estimation) for all parameters that maintain the optimal basis. This provides one single range of values of uncertainty for all parameters.

Parametric Analysis: Simultaneous changes of dependent parameter values from their nominal values that maintain the optimal basis. This provides the maximum magnitude of change for values of dependent parameters.

Sensitivity Analysis: One change at-a-time in any parameter value that maintains the optimal basis.

The Linear Programming Model

The LP model can be stated as:

$$\begin{aligned}
 & \text{Maximize } z = \sum_{j=1}^n c_j x_j \\
 & \text{st} \\
 & \sum_{j=1}^n a_{ij} x_j \leq b_i, i = 1, 2, \dots, m \\
 & \text{and } x_j \geq 0, j = 1, 2, \dots, n
 \end{aligned} \tag{1}$$

where c_j are the cost coefficients, b_i are the R.H.S. values with a_{ij} , as the technological constraints coefficients.

Discrete changes having sensitivity analysis as its investigation process for determining the new optimal solution when the original R.H.S. values (b_i) are replaced by new R.H.S. values \bar{b}_i . Much work has been done in this area. Continuous changes by which under parametric analysis we try to observe the transitions that occur as b_i is changed continuously into \bar{b}_i

$$b_i(\theta) = b_i + \alpha_i \theta \quad \theta \geq 0 \tag{2}$$

We in this paper carry out the parametric analysis of the b_i which are considered as parameters of the system.

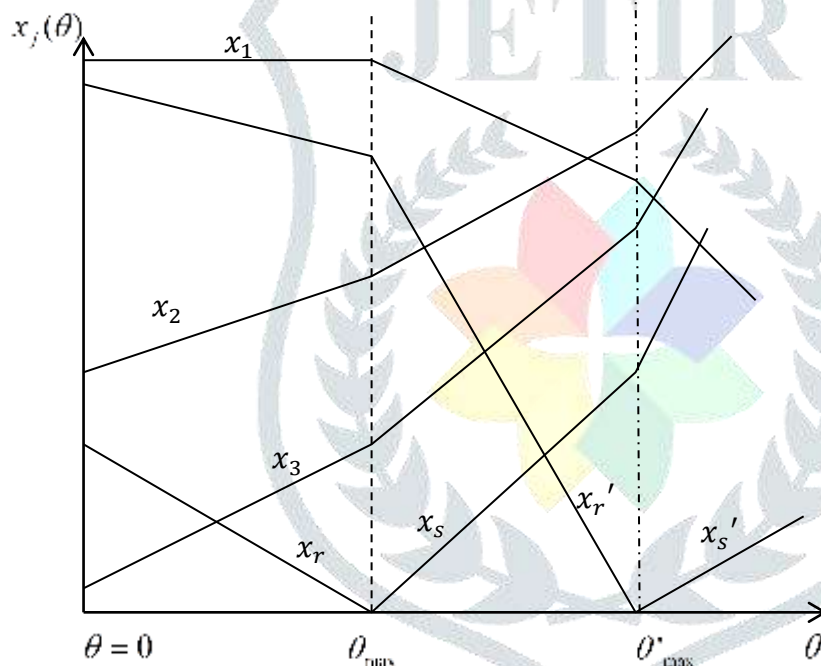


fig.1

As θ increases (i.e. $0 \leq \theta \leq \theta_{\max}$), the basic variables will change their values. Some will increase (e.g. x_r' and x_r), others will decrease (e.g. x_r and x_s) and some may not change at all (e.g. x_1). If $b_i(\theta) \geq 0, \forall_i$ the original basis remains optimal because the $(z_i - c_j)$ which are the simplex multipliers that determine the optimality condition depend only on the basis and not on the values of the basic variables, as long as feasibility is maintained. Eventually, however, a value $\theta = \theta_{\max}$ is reached at which a variable, say x_r hits zero and becomes negative for $\theta > \theta_{\max}$. If an optimal solution exists for $\theta = \theta_{\max} + \epsilon$ where ϵ is arbitrarily small, then the values of the basic variables for that optimal solution must join continuously with the values of the basic variables for the optimal solution valid for $\theta = \theta_{\max} - \epsilon$. This means that to push θ beyond θ_{\max} we must remove x_r (because removing any other variable would cause that variable to be discontinuous) and replace it by a nonbasic variable x_s , which is also zero at $\theta = \theta_{\max}$ but which increases in value as θ goes beyond θ_{\max} , and which maintains optimality. The new optimal basis obtained by replacing x_r with x_s will have its own θ range of validity given by

$$\theta_{\max} \leq \theta \leq \theta'_{\max}$$

This is shown in fig. 1.1 where another variable x_r' threatens to become negative.

We now state some lemma and their proofs which will enable us to assume and vividly comprehend some of the steps contained in this work (Ekoko, 2004).

Background of the Procedure

Assume that we have found an optimal solution for the original right-hand side b_i . If the b_i are to change continuously, then we can write for the current b_i

$$b_i(\theta) = b_i + \alpha_i \theta \quad \theta \geq 0 \quad \text{----- (2)}$$

if the changes are linearly related. The right-hand side is considered to be a linear function of a nonnegative parameter θ , and the α_i are constants which are the input data. In most problems, the changes in the b_i can be linearly related so that equation (2) will hold over the θ range of interest.

The α_i can be any numbers, positive or negative or zero. For example, if we wish to increase b_4 and keep other b_i constant, then $\alpha_4 = 1$ and all other $\alpha_i = 0$. Any other positive number for α_4 can also be used. If we wanted to increase b_5 and at the same time decrease b_6 at one third the rate, then $\alpha_6 = -\frac{1}{3}$, $\alpha_5 = 1$ and all other $\alpha_i = 0$. Again, any negative and positive numbers would do as well for α_6 and α_5 respectively as long as $-\frac{\alpha_6}{\alpha_5} = \frac{1}{3}$

Lemma 1: If the changes in the RHS are linearly related as stated in equation (2) with $\sum_{k=1}^m \beta_{ik} \alpha_k < 0$ for some or all i , then

$$\theta_{\max} = \min_i \frac{\sum_{k=1}^m \beta_{ik} b_k}{\left| \sum_{k=1}^m \beta_{ik} \alpha_k \right|}$$

Proof

Let the inverse matrix be

$$B^{-1} = \begin{bmatrix} \beta_{11} & \beta_{12} & \dots & \beta_{1m} \\ \beta_{21} & \beta_{22} & \dots & \beta_{2m} \\ \dots & \dots & \dots & \dots \\ \beta_{m1} & \beta_{m2} & \dots & \beta_{mm} \end{bmatrix}$$

The original optimum is given by:

$$x_i = \sum_{k=1}^m \beta_{ik} b_k, \quad i = 1, 2, \dots, m$$

and $z = \sum_{j=1}^m c_j x_j$ (3)

We know from equation (3), that the values of the basic variables for the original optimum are given by

$$x_i = \sum_{k=1}^m \beta_{ik} b_k, \quad i = 1, 2, \dots, m$$

where, as always, we assume that the variables are relabeled so that the optimal basis involves the first m variables. Since b_k is a function of θ , we obtain, by substituting equation (2) into equation (3),

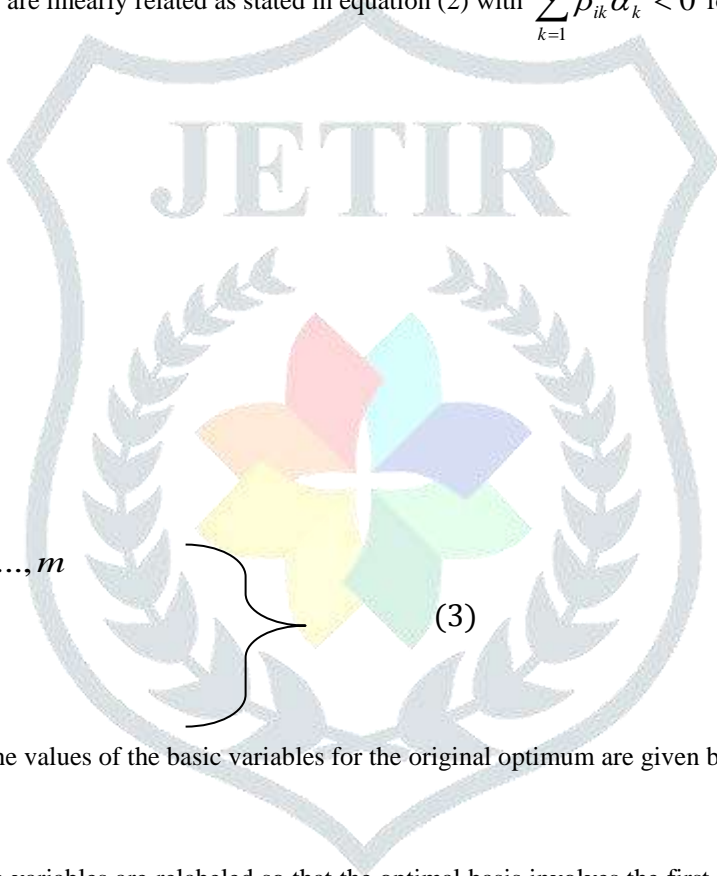
$$x_i(\theta) = \sum_{k=1}^m \beta_{ik} b_k + \theta \sum_{k=1}^m \beta_{ik} \alpha_k \quad \text{----- (4)}$$

As θ increases from zero value, the RHS values change correspondingly until the following all-inclusive situations arise:

- (a) $\sum_{k=1}^m \beta_{ik} b_k \geq 0$ for all i
- (b) $\sum_{k=1}^m \beta_{ik} \alpha_k < 0$ for some or all i

In (a) θ can take any positive value without violating feasibility condition. The obvious implication of this is that the optimal basis remains optimal no matter how large the changes are in the b_i (for the given α_i). In this case, we say that the RHS is open, and parametric analysis terminates because there is nothing left for it to do. In (b) however, we cannot increase θ indefinitely without violating feasibility condition. We have to determine the largest value that can be assigned to θ i.e. θ_{\max}

This is given by:



$$\theta_{max} = \min_i \frac{\sum_{k=1}^m \beta_{ik} b_k}{\sum_{k=1}^m \beta_{ik} \alpha_k} = \frac{\sum_{k=1}^m \beta_{ik} b_k}{\sum_{k=1}^m \beta_{ik} \alpha_k} \text{-----(5)}$$

where it is understood that we consider only those i for which

$$\sum_{k=1}^m \beta_{ik} \alpha_k < 0$$

The variable which goes to zero when $\theta = \theta_{max}$ is denoted by $i = r$. This variable must be removed from the basis, and another one must take its place if θ is to be pushed beyond θ_{max} . Our old optimal basis remained optimal up to this point.

Lemma 2

The new basis is formed when the non-basic variable x_s replaces the basic variable x_r in the interval $\theta_{max} \leq \theta \leq \theta'_{max}$ if

$$x_s(\theta) = \frac{b'_r(\theta)}{a'_{rs}}, \text{ then } \frac{(z_s - c_s)'}{|a'_{rs}|} \leq \frac{(z_j - c_j)'}{|a'_{rj}|}$$

Proof

We know from equation (5) which variable x_r to drop if $\theta = \theta_{max} + \epsilon$. The basis variable x_s to enter the basis should be such that it can maintain (i) feasibility and (ii) optimality.

(i) Feasibility Condition ($x_i \geq 0 \forall i$)

We know that if x_s replaces x_r in the interval $\theta_{max} \leq \theta \leq \theta'_{max}$, then the new value of x_s will be

$$x_s = \frac{b'_r}{a'_{rs}}$$

and this evidently still holds if b'_r and x_s are considered functions of θ so that

$$x_s(\theta) = \frac{b'_r(\theta)}{a'_{rs}}$$

In the interval $\theta_{max} \leq \theta \leq \theta'_{max}$ i.e. when $\theta = \theta'_{max} + \epsilon, b'_r(\theta) < 0$ and if $x_s(\theta) > 0$ then $a'_{rs} < 0$, if all $a'_{rj} \geq 0$, then we cannot increase θ beyond θ_{max} without violating feasibility and the process terminates.

Hence x_s is chosen from j columns that have $a'_{rj} < 0$

(ii) Optimality condition [the new simplex multipliers $(z_j - c_j)^* \geq 0 \forall j$]

By second part of Gauss-Jordan Elimination scheme,

$$(z_j - c_j)^* = (z_j - c_j)' - \frac{a'_{rj}(z_s - c_s)'}{a'_{rs}} \text{-----(6)}$$

and if our new basis is to be optimal, then s must be chosen so that

$$\left[(z_j - c_j)' - \frac{a'_{rj}(z_s - c_s)'}{a'_{rs}} \right] \geq 0 \forall j \text{-----(7)}$$

Since $a'_{rs} < 0$ as shown under feasibility condition, then equation (7) can be rewritten as:

$$\left((z_j - c_j)' + \frac{a'_{rj}(z_s - c_s)'}{|a'_{rs}|} \right) \geq 0 \text{-----(8)}$$

Equation (8) is certainly nonnegative for those j whose $a'_{rj} \geq 0$. If $a'_{rj} < 0$, then

$$\left((z_j - c_j)' + \frac{|a'_{rj}|(z_s - c_s)'}{|a'_{rs}|} \right) \geq 0 \text{ or } \frac{(z_j - c_j)'}{|a'_{rj}|} \geq \frac{(z_s - c_s)'}{|a'_{rs}|} \text{-----(9)}$$

And s is determined by

$$\frac{(z_s - c_s)'}{|a'_{rs}|} = \min_{a'_{rj} < 0} \frac{(z_j - c_j)'}{|a'_{rj}|}$$

Note that $a'_{rj} = \sum_{k=1}^m \beta_{rk} a_{kj}$

Numerical Illustration of sensitivity analysis

Maximize $z = 3x_1 + 2x_2 + 5x_3$
 s.t $x_1 + 2x_2 + x_3 \leq 430$
 $3x_1 + 2x_3 \leq 460$
 $x_1 + 4x_2 \leq 420$
 $x_1, x_2, x_3 \geq 0$

Minimize $z = 430y_1 + 460y_2 + 420y_3$
 Subject to $y_1 + 3y_2 + y_3 \geq 3$
 $2y_1 + 4y_3 \geq 2$
 $y_1 + 2y_2 \geq 5$
 $y_1, y_2, y_3 \geq 0$

The associated optimum tableau for the primal is given as

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	1350
x_2	$-\frac{1}{4}$	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	100
x_3	$\frac{3}{2}$	0	1	0	$\frac{1}{2}$	0	230
x_6	2	0	0	-2	1	1	20

Assuming the gaps are observed in the optimal solution tableau

Basic	x_1	x_2	x_3	x_4	x_5	x_6	Solution
z	4	0	0	1	2	0	d
x_2	a	1	0	$\frac{1}{2}$	$-\frac{1}{4}$	0	q
x_3	b	0	1	0	$\frac{1}{2}$	0	r
x_6	c	0	0	-2	1	1	s

The optimal inverse matrix is given as

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix}$$

To determine $a, b, c,$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{1}{4} \\ \frac{3}{2} \\ 2 \end{pmatrix}$$

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 430 \\ 460 \\ 420 \end{pmatrix} = \begin{pmatrix} 100 & q \\ 230 & r \\ 20 & s \end{pmatrix}$$

To determine d

$x_2 = q = 100$
 $x_3 = r = 230$
 $x_1 = 0$
 $d_0 = 3x_1 + 2x_2 + 5x_3$
 $d_0 = 1350$

Now suppose that changes are made in the capacities of the three constraints according to the following cases

- a. $\begin{pmatrix} 460 \\ 500 \\ 400 \end{pmatrix}$ b. $\begin{pmatrix} 500 \\ 400 \\ 600 \end{pmatrix}$ c. $\begin{pmatrix} 300 \\ 800 \\ 200 \end{pmatrix}$ d. $\begin{pmatrix} 450 \\ 700 \\ 350 \end{pmatrix}$

Determine the optimum solution in each case

(a)

$$\begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 460 \\ 500 \\ 400 \end{pmatrix} = \begin{pmatrix} 105 \\ 250 \\ -20 \end{pmatrix} \text{ Infeasible solution}$$

$x_1 = 0, x_2 = 105, x_3 = 250$
 Let the optimal solution be d_1
 $d_1 = 3(0) + 2(105) + 5(250)$
 $d_1 = 1460$

(b)

$$\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 500 \\ 400 \\ 600 \end{pmatrix} = \begin{pmatrix} 150 \\ 200 \\ 0 \end{pmatrix}$$

$x_1 = 0, x_2 = 105, x_3 = 250$
 $d_2 = 3(0) + 2(150) + 5(200)$

$d_2 = 1300$

(c)

$$\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 300 \\ 800 \\ 200 \end{pmatrix} = \begin{pmatrix} -50 \\ 400 \\ 400 \end{pmatrix}$$

$x_1 = 0, x_2 = -50, x_3 = 400$
 $d_3 = 3(0) + 2(-50) + 5(400)$
 $d_3 = 1900$

(d)

$$\begin{pmatrix} 1/2 & -1/4 & 0 \\ 0 & 1/2 & 0 \\ -2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 450 \\ 700 \\ 350 \end{pmatrix} = \begin{pmatrix} 50 \\ 350 \\ 0 \end{pmatrix}$$

$x_1 = 0, x_2 = 50, x_3 = 350$
 $d_4 = 3(0) + 2(50) + 5(350)$
 $d_4 = 1850$

d_n	x_1	x_2	x_3	S
1350	0	100	230	20
1460	0	105	250	-20
1300	0	150	200	0
1900	0	-50	400	400
1850	0	50	350	150

Numerical Illustration of Parametric Analysis

Changes in the constraints in real life LP problems are crucial, especially when we consider the fact that the quantity and quality of variables fluctuate from time to time and from season to season (Oluwatusin, 2013).

Example

$\max z = 2x_1 + 3x_2$

Subject to

$-x_1 + 2x_2 \leq 4$

$x_1 + x_2 \leq 6$

$x_1 + 3x_2 \geq 9$

$x_1, x_2 \geq 0$

(OPTIMUM)

B.V	$x_1 x_2 x_3 x_4 x_5$					R.H.S
x_2	0	1	0	-1/2	1/2	3/2
x_3	0	0	1	5/2	-3/2	11/2
x_1	1	0	0	3/2	-1/2	9/2
Z	0	0	0	3/2	1/2	27/2

Considering that we have found an optimal solution corresponding to the original right –hand side b_i , if the b_i are to change continuously, then we can write for the current \bar{b}_i as

$$\bar{b}_i(\theta) = b_i + \alpha_i \theta \quad \theta \geq 0 \text{ -----(2)}$$

If the changes are linearly related, the right –hand side is considered to be a linear functions of a non negative parameter θ , and the constants α_i are the input data. This is presented in equation (2)

Example 1

Let the input data be

$$\alpha_1 = 2, \alpha_2 = 1 \text{ and } \alpha_3 = 1$$

$$\bar{b} = \begin{pmatrix} b_1 + \alpha_1 \theta \\ b_2 + \alpha_2 \theta \\ b_3 + \alpha_3 \theta \end{pmatrix}$$

From Optimum Tableau

$$B^{-1} \alpha_k = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{5}{2} & -\frac{3}{2} \\ 0 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$$

Since $\sum_{k=1}^m B_{ik} \alpha_k \geq 0$ for i then θ_{max} is infinite because θ can be increased indefinitely without violating feasibility condition. The R.H.S is said to be OPEN and we need not to proceed further.

Example 2

Iteration 1 of parametric analysis

Let $\alpha_1 = 1, \alpha_2 = -0.5$ and $\alpha_3 = -2$ be the constants as input data for the parametric analysis.

From Optimum Tableau

$$\beta^{-1} \alpha_k = \begin{pmatrix} 0 & -\frac{1}{2} & \frac{1}{2} \\ 1 & \frac{5}{2} & -\frac{3}{2} \\ 0 & \frac{3}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -2 \end{pmatrix} = \begin{pmatrix} -\frac{3}{4} \\ \frac{11}{4} \\ \frac{1}{4} \end{pmatrix}$$

$$\text{Compute } \min_i \frac{\sum_{k=1}^m \beta_{ik} b_k}{\sum_{k=1}^m \beta_{ik} \alpha_k} = \min_i \left\{ \frac{3/2}{|-3/4|} \right\} = 2$$

Thus, row 1 is the pivot row

Not applicable here, since $\sum B_{rk} a_{kj}$ is not ≥ 0 for all i

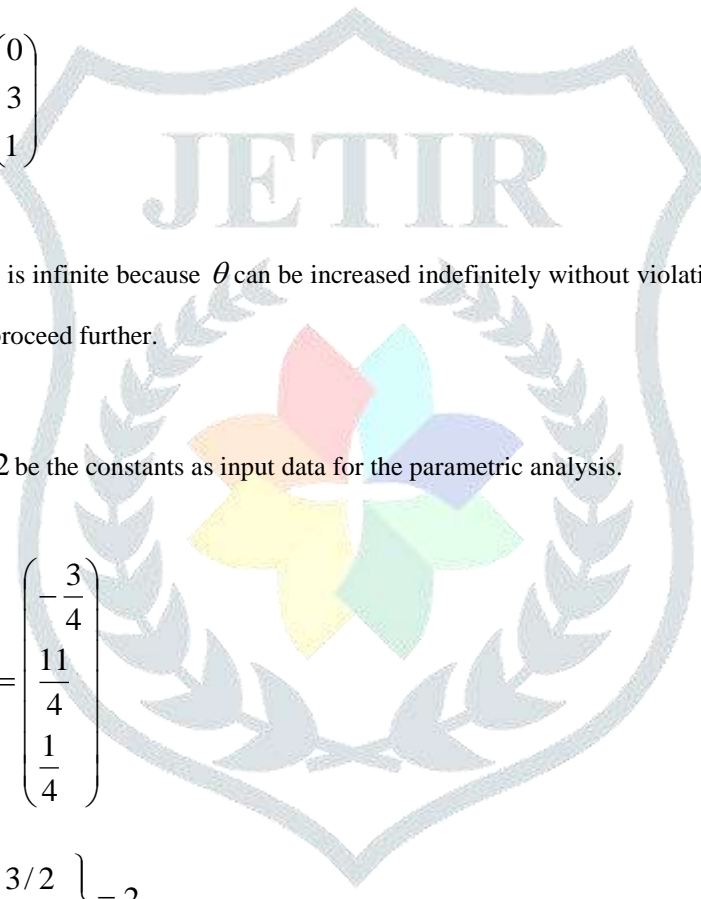
$$\alpha'_{rj} = \sum B_{rk} a_{kj} \text{ for } j = 1, 2, \dots, n$$

$\alpha'_j = \sum B_{rk} a_{kj}$ which is 1st row of parameter tableau

$$\alpha'_{ij} = \left(0, 1 \ 0 \quad -\frac{1}{2}, \frac{1}{2} \right)$$

Column selection

$$\text{Min}_{a_{ij} < 0} \frac{z_j - c_j}{|a_{ij}|} = \frac{3/2}{|-1/2|} = 3$$



Thus, $x_3 = x_s$ pivot column is the 4th column and $\left(-\frac{1}{2}\right)$ is our pivot element

$$x_i(t) = \sum_{k=1}^m B_{ik} b_k + \theta \sum_{k=1}^m B_{ik} a_k \quad \text{----- (3)}$$

From the optimal tableau and equation 3, we have
Parametric tableau 1

B.V	$x_1 x_2 x_3 x_4 x_5$					R.H.S
$x_2 x_3$	0	1	0	(-1/2)	1/2	0 →
x_1	0	0	1	5/2	-3/2	11
	1	0	0	3/2	-1/2	5
Z	0	0	0	3/2	1/2	27/2

Not applicable here

Current $\theta_{\max} = 2$ from determination of pivot row selection

2nd Iteration

Obtain parameter tableau 2

B.V	$x_1 x_2 x_3 x_4 x_5$					R.H.S
x_4	0	-2	0	1	-1	0
x_3	0	5	1	0	1	11
x_1	1	3	0	0	1	5
Z	0	3	0	0	2	27/2

Pivot row selection

$$\min_i \frac{\sum_{k=1}^m \beta_{ik} b_k}{\left| \sum_{k=1}^m \beta_{ik} \alpha_k \right|} = \min_i \left\{ \frac{13}{|-3/2|} \cdot \frac{9}{|-2|} \right\} = \min \{8.7, 4.5\} = 4.5$$

Thus, third row is the pivot row

Not applicable here

$$a'_{ij} = (1 \ 3 \ 0 \ 0 \ 1) \text{ i.e no } a'_{ij} \text{ is } < 0$$

Since $a_{ij} \geq 0$, we cannot increase θ beyond the current θ_{\max} without violation of feasibility condition. That is θ has reached a maximum possible value.

∴ $\theta_{\max} = 4.5$ as obtained on determination of pivot row of 2nd iteration

Not applicable

Example 3

Iteration 1

Let $\alpha_1 = 0.1, \alpha_2 = -0.2$ and $\alpha_3 = -0.3$ be the constants as input data for the parametric analysis.

Step 4:

$$\min_i \frac{\sum_{k=1}^3 \beta_{ik} b_k}{\left| \sum_{k=1}^3 \beta_{ik} \alpha_k \right|} = \min_i \left\{ \frac{3/2}{|-1/20|}, \frac{9/2}{|-3/20|} \right\}$$

$$= \min \{30, 30\} = 30$$

We arbitrarily select row 1 as pivot row

$$x_i(0) = \sum_{k=1}^3 B_{ik} b_k + \theta \sum_{k=1}^3 B_{ik} a_k$$

Not applicable here since $\sum B_{ij} \alpha_k$ is not ≥ 0 for all i

$$a'_{rj} = \sum_{k=1}^m B_{ik} a_{ij} \text{ which is 1st row of parametric tableau 5}$$

$$\begin{pmatrix} 0 \\ 0 \quad -\frac{1}{2} \quad \frac{1}{2} \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$a'_{ij} = (0 \quad 1 \quad 0 \quad -\frac{1}{2} \quad \frac{1}{2})$$

Parametric Tableau 3

B.V	x ₁	x ₂	x ₃	x ₄	x ₅	R.H.S
x ₂	0	1	0	(-1/2)	1/2	0
x ₃	0	0	1	5/2	-3/2	7
x ₁	1	0	0	3/2	-1/2	0 →
Z	0	0	0	3/2	1/2	27/2

Column selection

$$\text{Min}_{a_{rj} < 0} \frac{z_j - c_j}{|a_{rj}|} = \frac{3/2}{|-1/2|} = 3 \text{ i.e column 4 is the pivot column.}$$

1st Row and 4th column are selected. Therefore $-\frac{1}{2}$ becomes the pivot element

Note applicable here

$$\text{current } \theta_{\max} = 30$$

2nd iteration

Obtain parametric tableau 4

Parametric Tableau 4

B.V	x ₁	x ₂	x ₃	x ₄	x ₅	R.H.S
x ₄	0	-2	0	1	-1	0
x ₃	0	5	1	0	1	7
x ₁	1	3	0	0	1	0 →
Z	0	3	0	3/2	1/2	27/2

Pivot row selection

$$\min_i \frac{\sum_{k=1}^m \beta_{ik} b_k}{\sum_{k=1}^m \beta_{ik} \alpha_k} = \min_i \left\{ \frac{13}{|-1/5|}, \frac{9}{|-3/20|} \right\} = \min \{65, 30\} = 30$$

Not applicable here

Discussion

Parametric analysis test helps the modeler to understand the dynamics of a system by experimenting with a range of values which offers insights into behaviour of a system in extreme situations. Many parameters in system dynamic models represent quantities that are very difficult or even impossible to measure. By this paper it is possible to determine how “sensitive” a model is to sudden and continuous changes in the model. In this work, attention has been given to parameters which were performed as a series of tests in which the modeler sets different parameter values to see how a change in the parameter causes a change in the dynamic behavior of the system.

In example 1, the θ_{\max} is infinite as θ can be increased indefinitely without violating feasibility condition. In this case the right-hand-side (RHS) values are said to be open.

In the examples, some of the basic variables increase in value and others decrease as the parameter θ continuously increases from θ to θ_{\max} in a given parametric iteration. Among the basic variables that decrease as θ increases, one or more of them reduces to the value zero at the end of a given parametric iteration. Such a basic variable is the one in the pivot row that is replaced by the nonbasic variable in the pivot column. When there is more than one parametric iteration, each of the iterations is demarcated by the ordinate of its θ_{\max} . In other words each parametric iteration has its own θ_{\max} and by the workings, we arrive at the final θ_{\max} . Therefore, the lemma is true.

Conclusion

This paper work has enabled us to understand the dynamic behaviour of a linear program with respect to the range of values which certain decision variables can take without violating optimality and feasibility conditions.

Moreover, sensitivity and parametric analysis are useful to management with respect to decision variables that are critical control purpose and to determine range of values which certain decision variable could take on without affecting a change in the optimum solution, which could also enable managers to identify investment opportunities and could provide information to enhance a more efficient allocation of resources and risk management.

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