A GENERAL QUEUING MODEL WITH ARBITRARY SERVICE TIME DISTRIBUTION

Aparna Kushwaha[1], Dr. Anshula Pandey[2], Dr. Varun Kumar Kashyap*[3]
Research Scholar, Associate Professor, Assistant Professor,
Department of Statistics,
University of Lucknow, Lucknow-226007,India

Abstract: In this paper we have considered the queuing system with Poisson arrival and arbitrary service time distribution with mean service rate μ per unit of time, there is single service channel, system capacity is limited to N and the service discipline is FCFS. The steady state equations governing the queue and many characteristics for the modellike expected number of customers in the system, expected queue length etc. are obtained. Particular cases when service time follows Erlang, Lomax and Laplace distributions are considered and the steady state equations and various characteristics for the model are derived.

Index Term - Queueing system, arrival distribution, service time distribution (Erlang, Lomax and Laplace distribution), queue discipline.

I. INTRODUCTION

In our daily life routine, we avail several services for e.g. at a service station we may encounter situations and service related tasks such Customers waiting for check-out service in a supermarket/complex mall, Cars waiting for parking or at a stop light on a road crossing, Flights waiting for take-off or landing at an airport and Damaged machines waiting for repairing services.These situations have been very common phenomenon of waiting. The Pioneer of Queuing theory was Danish Mathematician A.K. Erlang (1909) who published "The Theory of Probability". The prodigious mathematician A.K. Erlang known as father of 'Teletraffic Theory' obtained the formulae related to traffic loads. The mathematical discussion on queuing theory substantially progressed in early 1930,s through the work of Pollaczk (1930, 1934), Kolmogorov (1931), Khintchine (1932, 1955), and others. Kendall (1951, 1953) gave a symmetric treatment of stochastic process occur the theory of Queues and Cox (1955) analyzed the congestion problems statistically. khintchine (1960) discussed mathematical methods in Queueing theory. Morse (1958) discussed many different kind of special Queuing problems in a wide way and Lee, A.M. (1958) gave applied queuing theory. An element of queuing theory with applications was given by T.L. Saaty (1961). Solution of certain typical problems in queuing and scheduling theory has been given by Arpana Badoni (2001).

Steady state difference equation

Let $P_n(t)$ is probability of n units in the system at time t and $P_n(t+\Delta t)$ is the probability of n units in the system at time $(t+\Delta t)$ then,

$$Pn(t+\Delta t) = P_n(t).P[\text{no arrival in the system during } \Delta t].P[\text{no service during } \Delta t] + P_{n-1}(t).P[1 \text{ arrival during } \Delta t].P[\text{no service during } \Delta t] + P_{n+1(t)}(t).P[\text{no arrival during } \Delta t].P[1 \text{ service during } \Delta t] + O(\Delta t); n \geq 1$$

{Since events are mutually exclusive & exhaustive and arrivals & services are independently distributed}

$$P_n(t+\Delta t) = P_n(t)[1-\lambda\Delta t + O(\Delta t)][1-\mu\Delta t + O(\Delta t)] + P_{n-1}(t)[\lambda\Delta t + O(\Delta t)][1-\mu\Delta t O(\Delta t)] + P_{n+1}(t)[1-\lambda\Delta t + O(\Delta t)][\mu\Delta t + O(\Delta t)] + O(\Delta t) \quad ; 1 \leq n \leq N-1$$

$$P_n(t+\Delta t) = P_n(t) - (\lambda+\mu)P_n(t)\Delta t + \lambda P_{n-1}(t)\Delta t + \mu(t)\Delta t + O(\Delta t) \quad ; 1 \leq n \leq N-1$$

{Combining all the terms of O(Δt)}

$$\lim_{\Delta t \to 0} \frac{P_n(t+\Delta t) - P_n(t)}{\Delta t} = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t}$$

$$P_n'(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) \quad ; 1 \leq n \leq N-1 \qquad \ldots (3.1)$$

Since $\lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t} \to 0$

For n = 0,

$$P_0(t+\Delta t) = P_0(t)[\text{no arrival during } \Delta t] + P_1(t).P[\text{no arrival during } \Delta t].P[1 \text{ service during } \Delta t] + O(\Delta t)$$

{Combining all the terms of O(Δt)}

$$P_0(t+\Delta t) = P_0(t)[1-\lambda\Delta t + O(\Delta t)] + P_1(t)[1-\lambda\Delta t + O(\Delta t)][\mu\Delta t + O(\Delta t)] + O(\Delta t)$$
$$= P_0(t) - \lambda P_0(t)\Delta t + \mu P_1(t)\Delta t + O(\Delta t)$$

{Combining all the terms of O(Δt)}

Or; $\lim_{\Delta t \to 0} \frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) + \lim_{\Delta t \to} \frac{O(\Delta t)}{\Delta t}$

Or; $P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$ ….(3.2)

and for n = N

$$P_N(t+\Delta t) = P[N \text{ units in the system at time } (t+\Delta t)]$$

$= P_N(t).\ P[\text{no service during } \Delta t] + P_{N-1}(t).\ P[1 \text{ arrival during } \Delta t].\ P[\text{no service during } \Delta t] + O(\Delta t)$

{Since for n = N, arrival rate λ = 0}

$= P_N(t)P\,[1 - \mu\Delta t + O(\Delta t)] + P_{N-1}(t)[\lambda\Delta t + O(\Delta t)][1 - \mu\Delta t + O(\Delta t)] + O(\Delta t)$

$P_N(t+\Delta t) = P_N(t) - \mu P_N(t)\Delta t + \lambda P_{N-1}(t)\Delta t + O(\Delta t)$

$$\lim_{\Delta t \to 0} \frac{P_N(t+\Delta t) - P_N(t)}{\Delta t} = -\mu P_N(t) + \lambda P_{N-1}(t) + \lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t}$$

Or; $P_N'(t) = -\mu P_N(t) + \lambda P_{N-1}(t)$ ….(3.3)

Under steady state conditions, $P_n(t) \to P_n$ for large t and $P_n'(t) \to 0$ ∀ n.

Thus the equations (3.1), (3.2) and (3.3) become

$0 = -(\lambda + \mu)P_n + \lambda P_{n-1} + \mu P_{n+1}$

Or; $P_{n+1} = \frac{1}{\mu}[(\lambda + \mu)P_n - \lambda P_{n-1}]$; for $1 \le n \le N-1$ ….(3.4)

$P_1 = \frac{\lambda}{\mu} P_0$ …. (3.5)

and, $P_N = \frac{\lambda}{\mu} P_{N-1}$ ….(3.6)

Putting n = 1 in equation (3.4), we get

$P_2 = \frac{1}{\mu}[(\lambda + \mu)P_1 - \lambda P_0]$

$= \frac{1}{\mu}\left[(\lambda + \mu)\frac{\lambda}{\mu} - \lambda\right] P_0$

$= \left(\frac{\lambda}{\mu}\right)^2 P_0$

Putting n = 2 in eq. (3.4), we get

$P_3 = \frac{1}{\mu}[(\lambda + \mu)P_2 - \lambda P_1]$

$= \left(\frac{\lambda}{\mu}\right)^3 P_0$ …………….

Putting n = (N−2), we get

$P_{N-1} = \frac{1}{\mu}[(\lambda + \mu)P_{N-2} - \lambda P_{N-3}]$

$= \frac{1}{\mu}\left[(\lambda + \mu)\left(\frac{\lambda}{\mu}\right)^{N-2} - (\lambda)\left(\frac{\lambda}{\mu}\right)^{N-3}\right] P_o$

$= \left(\frac{\lambda}{\mu}\right)^{N-1} P_0$

Therefore, $P_N = \left(\frac{\lambda}{\mu}\right) P_{N-1}$

$= \frac{\lambda}{\mu}.\left(\frac{\lambda}{\mu}\right)^{N-1} P_0$ …. (3.7)

Hence in general, $P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$; n = 0,1,………….N

And $P_{n+1} = 0$; for $n > N$, *as the capacity of the system is limited to N*

For finding the value of P_0,

Since; $\sum_{n=0}^{N} P_n = 1$

Or; $P_0 + P_1 + \ldots\ldots\ldots. P_N = 1$

Or; $\left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \cdots + \left(\frac{\lambda}{\mu}\right)^N\right] P_0 = 1$

Or; $\frac{1\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)\right\}} P_0 = 1$ {sum of (N+1) terms of G.P with common ratio $\frac{\lambda}{\mu}$}

$$\text{Or; } P_o = \begin{cases} \frac{\left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}} & ; \quad \frac{\lambda}{\mu} \neq 1 \\ \frac{1}{N+1} & ; \quad \frac{\lambda}{\mu} = 1 \end{cases} \quad \ldots. (3.8)$$

Thus the steady state equation is given by

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \left\{\frac{\left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}\right\} & ; \frac{\lambda}{\mu} \neq 1 \\ \frac{1}{N+1} & ; \frac{\lambda}{\mu} = 1 \end{cases} \quad ; 0 \le n \le N \quad \ldots. (3.9)$$

Characteristics of the model

1. Probability that queue length ≥ k

$P[\text{queue Size} \geq K] = \sum_{n=k}^{N} P_n$

$=\sum_{n=k}^{N} P_n - \sum_{n=0}^{K-1} P_n$

$= 1 - \sum_{n=0}^{K-1} \left(\frac{\lambda}{\mu}\right)^n P_0$

$= 1 - P_0 \sum_{n=0}^{K-1} \left(\frac{\lambda}{\mu}\right)^n$

$= 1 - P_0 \left[\frac{\left\{1-\left(\frac{\lambda}{\mu}\right)^k\right\}}{1-\frac{\lambda}{\mu}}\right]$ {Sum of k terms of G.P. with common ration $\frac{\lambda}{\mu}$; $\frac{\lambda}{\mu} \neq 1$}

$= 1 - \frac{\left(1-\frac{\lambda}{\mu}\right)\left\{1-\left(\frac{\lambda}{\mu}\right)^k\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}\left\{1-\frac{\lambda}{\mu}\right\}}$; $\frac{\lambda}{\mu} \neq 1$

$= 1 - \frac{1-\left(\frac{\lambda}{\mu}\right)^k}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}$; $\frac{\lambda}{\mu} \neq 1$

$= \frac{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\} - \left\{1-\left(\frac{\lambda}{\mu}\right)^k\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}} = \frac{\left(\frac{\lambda}{\mu}\right)^k - \left(\frac{\lambda}{\mu}\right)^{N+1}}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}$

$= \frac{\left(\frac{\lambda}{\mu}\right)^k . \left[1-\left(\frac{\lambda}{\mu}\right)^{N-k+1}\right]}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}$; $\frac{\lambda}{\mu} \neq 1$ …. (3.10)

2. Average number of customers is the system is given by

$$E(n) = \sum_{n=0}^{N} P_n = \sum_{n=0}^{N} n.\left(\frac{\lambda}{\mu}\right)^n P_0$$

$= P_0 . S$; Where $S = \sum_{n=0}^{N} n.\left(\frac{\lambda}{\mu}\right)^n$

Or; $S = \left(\frac{\lambda}{\mu}\right) + 2\left(\frac{\lambda}{\mu}\right)^2 + 3\left(\frac{\lambda}{\mu}\right)^3 + \cdots + N\left(\frac{\lambda}{\mu}\right)^N$

$\frac{\lambda}{\mu} S = \left(\frac{\lambda}{\mu}\right)^2 + 2\left(\frac{\lambda}{\mu}\right)^3 + \cdots + (N-1)\left(\frac{\lambda}{\mu}\right)^N + N\left(\frac{\lambda}{\mu}\right)^{N+1}$

$$(1-\frac{\lambda}{\mu})S = \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \cdots + \left(\frac{\lambda}{\mu}\right)^N - N\left(\frac{\lambda}{\mu}\right)^{N+1}$$

$= \frac{\left(\frac{\lambda}{\mu}\right)\left\{1-\left(\frac{\lambda}{\mu}\right)^N\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)\right\}} - N\left(\frac{\lambda}{\mu}\right)^{N+1}$

$= \frac{\frac{\lambda}{\mu}\left\{\left(1-\left(\frac{\lambda}{\mu}\right)^N\right) - N\left(1-\frac{\lambda}{\mu}\right)\left(\frac{\lambda}{\mu}\right)^N\right\}}{\left(1-\frac{\lambda}{\mu}\right)}$

$= \frac{\left(\frac{\lambda}{\mu}\right)\left\{1-(N+1)\left(\frac{\lambda}{\mu}\right)^N + N\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}{\left(1-\frac{\lambda}{\mu}\right)}$

or; $S = \frac{\left(\frac{\lambda}{\mu}\right)\left[1-(N+1)\left(\frac{\lambda}{\mu}\right)^N + N\left(\frac{\lambda}{\mu}\right)^{N+1}\right]}{\left(1-\frac{\lambda}{\mu}\right)^2}$; $\frac{\lambda}{\mu} \neq 1$

Therefore, $E(n) = P_0 . S$

$= \frac{\left(1-\frac{\lambda}{\mu}\right)}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}} . \frac{\frac{\lambda}{\mu}\left[1-(N+1)\left(\frac{\lambda}{\mu}\right)^N + N\left(\frac{\lambda}{\mu}\right)^{N+1}\right]}{\left\{1-\frac{\lambda}{\mu}\right\}^2}$

$E(n) = \frac{\frac{\lambda}{\mu}\left[1-(N+1)\left(\frac{\lambda}{\mu}\right)^N + N\left(\frac{\lambda}{\mu}\right)^{N+1}\right]}{\left\{1-\frac{\lambda}{\mu}\right\}\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}$; $\frac{\lambda}{\mu} \neq 1$ …. (3.11)

3. Average queue length is given by E(m) ; where m = (n-1)

i.e. $L_q = E(m) = \sum_{n=1}^{N}(n-1) P_n$

$= \sum_{n=0}^{N} nP_n - \sum_{n=1}^{N} P_n - P_0 + P_0$

$= E(n) - \sum_{n=0}^{N} P_n - P_0$

Or; $E(m) = E(n) - \{1 - P_0\}$

$$= E(n) - \left[1 - \frac{\left(1-\frac{\lambda}{\mu}\right)}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}\right] \qquad ; \frac{\lambda}{\mu} \neq 1$$

$$= E(n) - \left[\frac{1-\left(\frac{\lambda}{\mu}\right)^{N+1} - \left(1-\frac{\lambda}{\mu}\right)}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}\right]$$

$$= E(n) - \left(\frac{\lambda}{\mu}\right)\left[\frac{1-\left(\frac{\lambda}{\mu}\right)^{N}}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}\right] \qquad ; \frac{\lambda}{\mu} \neq 1$$

$$= \left(\frac{\lambda}{\mu}\right)\frac{\left[1-(N+1)\left(\frac{\lambda}{\mu}\right)^{N} + N\left(\frac{\lambda}{\mu}\right)^{N+1}\right]}{\left\{1-\frac{\lambda}{\mu}\right\}\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}} - \left(\frac{\lambda}{\mu}\right)\left[\frac{1-\left(\frac{\lambda}{\mu}\right)^{N}}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}\right]$$

$$= \left(\frac{\lambda}{\mu}\right)\frac{\left[\left\{1-(N+1)\left(\frac{\lambda}{\mu}\right)^{N} + N\left(\frac{\lambda}{\mu}\right)^{N+1} - \left\{1-\frac{\lambda}{\mu}\right\}\left\{1-\left(\frac{\lambda}{\mu}\right)^{N}\right\}\right]}{\left(1-\frac{\lambda}{\mu}\right)\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}$$

$$= \left(\frac{\lambda}{\mu}\right)\frac{\left[\left\{1-(N+1)\left(\frac{\lambda}{\mu}\right)^{N} + N\left(\frac{\lambda}{\mu}\right)^{N+1} - 1 + \left(\frac{\lambda}{\mu}\right)^{N} + \left(\frac{\lambda}{\mu}\right) - \left(\frac{\lambda}{\mu}\right)^{N+1}\right]}{\left(1-\frac{\lambda}{\mu}\right)\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)\left[\left(\frac{\lambda}{\mu}\right) - N\left(\frac{\lambda}{\mu}\right)^{N} + (N-1)\left(\frac{\lambda}{\mu}\right)^{N+1}\right]}{\left(1-\frac{\lambda}{\mu}\right)\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}$$

$$\text{Or;} \quad E(m) = \frac{\left(\frac{\lambda}{\mu}\right)^{2}\left[1-N\left(\frac{\lambda}{\mu}\right)^{N-1} + (N-1)\left(\frac{\lambda}{\mu}\right)^{N}\right]}{\left\{1-\left(\frac{\lambda}{\mu}\right)\right\}\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}} \qquad \text{....(3.12)}$$

4. Average length of non-empty queue is given by,

$E[m/m>0] = \frac{E(m)}{P(m>0)}$, $m = (n-1)$

$P[m>0] = P(n-1>0)$

$= 1 - P(n\leq 1)$

$= 1 - P_0 - P_1$

$$= 1 - \frac{1-\left\{\left(\frac{\lambda}{\mu}\right)\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}} - \frac{\left\{1-\left(\frac{\lambda}{\mu}\right)\right\}\left\{\frac{\lambda}{\mu}\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}} \qquad ; \frac{\lambda}{\mu} \neq 1$$

$$= \frac{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\} - \left\{1-\frac{\lambda}{\mu}\right\} - \left\{\left(\frac{\lambda}{\mu}\right) - \left(\frac{\lambda}{\mu}\right)^{2}\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}} ; \frac{\lambda}{\mu} \neq 1$$

$$= \frac{\left(\frac{\lambda}{\mu}\right)^{2}\left\{1-\left(\frac{\lambda}{\mu}\right)^{N-1}\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}$$

$$E[m/m>0] = \frac{\dfrac{\left(\frac{\lambda}{\mu}\right)^{2}\left\{1-\left(\frac{\lambda}{\mu}\right)^{N-1} + (N-1)\left(\frac{\lambda}{\mu}\right)^{N}\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)\right\}\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}}{\dfrac{\left(\frac{\lambda}{\mu}\right)^{2}\left\{1-\left(\frac{\lambda}{\mu}\right)^{N-1}\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}}$$

Thus, the average length of non-empty queue is given by;

$$E[m/m>0] = \frac{\left\{1-N\left(\frac{\lambda}{\mu}\right)^{N-1} + (N-1)\left(\frac{\lambda}{\mu}\right)^{N}\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)\right\}\left\{1-\left(\frac{\lambda}{\mu}\right)^{N-1}\right\}} \qquad \text{....(3.13)}$$

Particular Cases

Case I.

3.1 Let the service time follows Erlang distribution with parameters (α, β) and its probability density function is given by

$$f(t) = \frac{\beta^{\alpha}}{(\alpha-1)!} t^{\alpha-1} e^{-\beta t} ; t \geq 0 \quad ; \ (\alpha, \beta) > 0 \qquad \text{....(3.1.1)}$$

With mean service rate

$\mu = E(t) = \frac{\alpha}{\beta}$

Putting $\mu = \frac{\alpha}{\beta}$ in equation (3.9), the steady state equation is given by

$$P_n = \begin{cases} \left(\frac{\lambda\beta}{\alpha}\right)^n \frac{\left\{1-\frac{\lambda\beta}{\alpha}\right\}}{\left\{1-\left(\frac{\lambda\beta}{\alpha}\right)^{N+1}\right\}} & ;\frac{\lambda\beta}{\alpha} \neq 1 \\ \frac{1}{N+1} & ;\frac{\lambda\beta}{\alpha} = 1 \end{cases} \quad \text{for } 0 \le n \le N \qquad ....(3.1.2)$$

Characteristics of the model

1)Probability that queue length ≥ k

$$P[\text{ queue size} \ge k] = \frac{\left(\frac{\lambda}{\mu}\right)^k\left[1-\left(\frac{\lambda}{\mu}\right)^{N-k+1}\right]}{\left[1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right]} \quad ;\frac{\lambda}{\mu} \neq 1$$

Putting $\mu = \frac{\alpha}{\beta}$, we get;

$$P[\text{queue size} \ge k] = \frac{\left(\frac{\lambda\beta}{\alpha}\right)^k\left[1-\left(\frac{\lambda\beta}{\alpha}\right)^{N-k+1}\right]}{\left[1-\left(\frac{\lambda\beta}{\alpha}\right)^{N+1}\right]} \quad ;\frac{\lambda\beta}{\alpha} \neq 1 \qquad ....(3.1.3)$$

2) Expected number of customers in the system is given by

$$E(n) = \frac{\left(\frac{\lambda}{\mu}\right)\left[1-(N+1)\left(\frac{\lambda}{\mu}\right)^N + N\left(\frac{\lambda}{\mu}\right)^{N+1}\right]}{\left(1-\frac{\lambda}{\mu}\right)\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}$$

$$= \frac{\left(\frac{\lambda\beta}{\alpha}\right)\left[1-(N+1)\left(\frac{\lambda\beta}{\alpha}\right)^N + N\left(\frac{\lambda\beta}{\alpha}\right)^{N+1}\right]}{\left(1-\frac{\lambda\beta}{\alpha}\right)\left\{1-\left(\frac{\lambda\beta}{\alpha}\right)^{N+1}\right\}} \quad ;\frac{\lambda\beta}{\alpha} \neq 1 \qquad ....(3.1.4)$$

3) Average queue length is given by

$$L_q = E(m) = \frac{\left(\frac{\lambda}{\mu}\right)^2\left[1-N\left(\frac{\lambda}{\mu}\right)^{N-1} + (N-1)\left(\frac{\lambda}{\mu}\right)^N\right]}{\left(1-\frac{\lambda}{\mu}\right)\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}$$

$$= \frac{\left(\frac{\lambda\beta}{\alpha}\right)^2\left[1-N\left(\frac{\lambda\beta}{\alpha}\right)^{N-1} + (N-1)\left(\frac{\lambda\beta}{\alpha}\right)^N\right]}{\left\{\left(1-\frac{\lambda\beta}{\alpha}\right)\right\}\left\{1-\left(\frac{\lambda\beta}{\alpha}\right)^{N+1}\right\}} \quad ;\frac{\lambda\beta}{\alpha} \neq 1 \qquad .... (3.1.5)$$

4) Average length of non-empty queue is given by,

$$E[m/m>0] = \frac{\left\{1-N\left(\frac{\lambda\beta}{\alpha}\right)^{N-1} + (N-1)\left(\frac{\lambda\beta}{\alpha}\right)^N\right\}}{\left\{\left(1-\frac{\lambda\beta}{\alpha}\right)\right\}\left\{1-\left(\frac{\lambda\beta}{\alpha}\right)^{N-1}\right\}} \quad ;\frac{\lambda\beta}{\alpha} \neq 1 \qquad ....(3.1.6)$$

Particularly for α = 1, all the results are same as those when the service time distribution is exponential with mean service rate μ per unit of time.

Case II.

3.2 Let the service time follows Lomax distribution with parameters (α, β) and its p.d.f. is given by:

$$f(t) = \frac{\beta}{\alpha}\left[1+\frac{t}{\alpha}\right]^{-(\beta+1)} \quad ; t > 0 \;\; ; (\propto, \beta) > 0$$

with mean service rate $\mu = E(t)$

$\mu = E(t) = \frac{\beta}{\alpha-1}$; for α >1

Putting $\mu = \frac{\beta}{\alpha-1}$ in (3.9), the steady state equation is given by

$$P_n = \begin{cases} \left(\frac{\lambda}{\mu}\right)^n \left\{\frac{\left(1-\frac{\lambda}{\mu}\right)}{1-\left(\frac{\lambda}{\mu}\right)^{N+1}}\right\} & ;\frac{\lambda}{\mu} \neq 1 \\ \frac{1}{N+1} & ;\frac{\lambda}{\mu} = 1 \end{cases} \quad ; 0 \le n \le N$$

$$P_n = \begin{cases} \frac{\left(\frac{\lambda(\alpha-1)}{\beta}\right)^n\left\{1-\frac{\lambda(\alpha-1)}{\beta}\right\}}{\left\{1-\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N+1}\right\}} & ;\frac{\lambda(\alpha-1)}{\beta} \neq 1 \\ \frac{1}{N+1} & ;\frac{\lambda(\alpha-1)}{\beta} = 1 \end{cases} \quad \text{For } 0 \le n \le N \qquad ....(3.2.1)$$

Characteristics of the model

1) **Probability that queue length ≥ k**

$$\text{Thus, } P[\text{queue Size} \ge k] = \frac{\left(\frac{\lambda}{\mu}\right)^k\left[1-\left(\frac{\lambda}{\mu}\right)^{N-k+1}\right]}{\left[1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right]} \quad ;\frac{\lambda}{\mu} \neq 1$$

$$P[\text{queue size} \ge k] = \frac{\left(\frac{\lambda(\alpha-1)}{\beta}\right)^k\left[1-\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N-k+1}\right]}{\left[1-\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N+1}\right]} \quad ;\frac{\lambda(\alpha-1)}{\beta} \neq 1 \qquad ....(3.2.2)$$

2) Expected number of customers in the system is given by

$$E(n) = \frac{\frac{\lambda(\alpha-1)}{\beta}\left[1-(N+1)\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N}+N\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N+1}\right]}{\left\{1-\frac{\lambda(\alpha-1)}{\beta}\right\}\left\{1-\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N+1}\right\}} \quad ; \frac{\lambda(\alpha-1)}{\beta} \neq 1 \qquad \text{….(3.2.3)}$$

3) Average queue length is given by;

$L_q = E(m)$

$$E(m) = \frac{\left(\frac{\lambda}{\mu}\right)^2\left[1-N\left(\frac{\lambda}{\mu}\right)^{N-1}+(N-1)\left(\frac{\lambda}{\mu}\right)^{N}\right]}{\left\{1-\left(\frac{\lambda}{\mu}\right)\right\}\left\{1-\left(\frac{\lambda}{\mu}\right)^{N+1}\right\}}$$

$$= \frac{\left(\frac{\lambda(\alpha-1)}{\beta}\right)^2\left[1-N\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N-1}+(N-1)\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N}\right]}{\left\{1-\left(\frac{\lambda(\alpha-1)}{\beta}\right)\right\}\left\{1-\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N+1}\right\}} \; ; \frac{\lambda(\alpha-1)}{\beta} \neq 1 \qquad \text{….(3.2.4)}$$

4) Average length of non-empty queue is given by

$$E(m/m>0) = \frac{\left\{1-N\left(\frac{\lambda}{\mu}\right)^{N-1}+(N-1)\left(\frac{\lambda}{\mu}\right)^{N}\right\}}{\left\{1-\left(\frac{\lambda}{\mu}\right)\right\}\left\{1-\left(\frac{\lambda}{\mu}\right)^{N-1}\right\}} \qquad ;\frac{\lambda}{\mu} \neq 1$$

$$= \frac{\left\{1-N\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N-1}+(N-1)\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N}\right\}}{\left\{1-\left(\frac{\lambda(\alpha-1)}{\beta}\right)\right\}\left\{1-\left(\frac{\lambda(\alpha-1)}{\beta}\right)^{N-1}\right\}} \qquad ;\frac{\lambda(\alpha-1)}{\beta} \neq 1 \qquad \text{….(3.2.5)}$$

For $\beta = 1$, all the results are same as those when the service time distribution is exponential.

Case III.

3.3 Let the service time't' follows Laplace distribution with parameters (α, β) i.e. its probability density function (pdf) is

$$f(t) = \frac{1}{2\beta}\exp\left(-\frac{|t-\alpha|}{\beta}\right) \qquad ; -\infty < t < \infty \;\; ; \; (\alpha, \beta) > 0 \qquad \text{….(3.3.1)}$$

$$= \frac{1}{2\beta}\begin{cases} \exp\left(-\frac{\alpha-t}{\beta}\right); & \text{if } t < \mu \\ \exp\left(-\frac{t-\alpha}{\beta}\right); & \text{if } t \geq \mu \end{cases}$$

Here, $\alpha > 0$ is a location parameter and $\beta > 0$, is a scale parameter, which is sometimes referred to as the diversity.

Mean service rate $\mu = E(t)$

$\mu = E(t) = \alpha$

Putting $\mu = \alpha$in (3.9), the steady state equation is given by

$$P_n = \begin{cases} \left(\frac{\lambda}{\alpha}\right)^n\left\{\frac{\left(1-\frac{\lambda}{\alpha}\right)}{1-\left(\frac{\lambda}{\alpha}\right)^{N+1}}\right\} & ;\frac{\lambda}{\alpha} \neq 1 \\ \frac{1}{N+1} & ;\frac{\lambda}{\alpha} = 1 \end{cases} \qquad ; 0 \leq n \leq N \qquad \text{….(3.3.2)}$$

Characteristics of the model

1)Probability that queue length ≥ k

$$P[\text{queue Size} \geq k] = \frac{\left(\frac{\lambda}{\alpha}\right)^k\left[1-\left(\frac{\lambda}{\alpha}\right)^{N-k+1}\right]}{\left[1-\left(\frac{\lambda}{\alpha}\right)^{N+1}\right]} \; ;\frac{\lambda}{\alpha} \neq 1 \qquad \text{….(3.3.3)}$$

2) Expected number of customers in the system is given by

$$E(n) = \frac{\frac{\lambda}{\alpha}\left[1-(N+1)\left(\frac{\lambda}{\alpha}\right)^{N}+N\left(\frac{\lambda}{\alpha}\right)^{N+1}\right]}{\left\{1-\frac{\lambda}{\alpha}\right\}\left\{1-\left(\frac{\lambda}{\alpha}\right)^{N+1}\right\}} \quad ; \frac{\lambda}{\alpha} \neq 1 \qquad \text{….(3.3.4)}$$

3) Average queue length is given by;

$L_q = E(m)$

$$E(m) = \frac{\left(\frac{\lambda}{\alpha}\right)^2\left[1-N\left(\frac{\lambda}{\alpha}\right)^{N-1}+(N-1)\left(\frac{\lambda}{\alpha}\right)^{N}\right]}{\left\{1-\left(\frac{\lambda}{\alpha}\right)\right\}\left\{1-\left(\frac{\lambda}{\alpha}\right)^{N+1}\right\}} \qquad ;\frac{\lambda}{\alpha} \neq 1 \qquad \text{….(3.3.5)}$$

4) Average length of non-empty queue is given by

$$E(m/m>0) = \frac{\left\{1-N\left(\frac{\lambda}{\alpha}\right)^{N-1}+(N-1)\left(\frac{\lambda}{\alpha}\right)^{N}\right\}}{\left\{1-\left(\frac{\lambda}{\alpha}\right)\right\}\left\{1-\left(\frac{\lambda}{\alpha}\right)^{N-1}\right\}} \qquad ;\frac{\lambda}{\alpha} \neq 1 \qquad \text{….(3.3.6)}$$

For $\alpha = 1$, all the results are same as those when the service time distribution is exponential.

REFERENCES

[1] Aftab Begum and D. Maheswari (2002): The M/M/c interdependent queueing model with controllable arrival rates, Opsearch, Vol. 39, No. 2, pp-89-110.

[2] Ancker, C. J. Jr. And Gaffarin, A. V. (1963): Some queueing problem with balking and reneging, Oper. Research. II, 928-937.

[3] Ancker, C. J. Jr. And Gaffarin, A. V. (1961): Queueing with reneging and multiple heterogeneous servers, SP. 372, system Dev. Corporation, pp. 145-151.

[4] Artalejo, A. R. and Lopez Herrero, M.J. (2000): On the single server retrial queue with balking, INFOR, Vol. 38, No. 1 pp. 33-50.

[5] Askin, O. Z. and Maker, P. T. (2001): Modeling a phone centre analysis of a multi- channel, Multi- resources processor sharing loss system Management School Vol-No. 4 pp. 605-610.

[6] Banal, Nikhil, (2003): Analysis of the M/G/1 processor- sharing- queue with bulk arrivals, O. R. Letters, 31, pp. 401-405.