MATRIX REPRESENTATION OF MONOTONE CONVOLUTION

Dr. Arun Dayal

Deptt. Of Mathematics, D.P.C.S.S.(Miller) Sr. Sec. School, Patna.

ABSTRACT

The monotone convolution of atomic measures is applied to the monotone product of matrix algebras. We prove here that the monotone convolution of $m \times m$ matrix and $n \times n$ marix becomes $mn \times mn$ matrix, which is a consequence of the algebraic construction of monotone product. We derive properties of atoms in the monotone convolution of atomic measures. We establish how moments change under the monotone convolution. We drive a differential equation about the minimum of support of a monotone convolution semigroup. We examine here that a property of a monotone convolution semigroup changes with respect to time parameter. Time-independent property is a property of a convolution semigroup. We derive that properties of non-commutative probability theory are time-independent. These properties are also time-independent in classical convolution semigroups. We classify strictly $\triangleright$-stable distributions or equivalently, $\triangleright$-infinitely divisible and self-similar distributions and show its equivalence to the free and Boolean cases.

KEY WORDS

Monotone convolution, Convolution semigroups, Probability measures, Central measure, Symmetric measure, Gaussian measure, Cauchy transform.

1. INTRODUCTION

Muraki indroduced the notion of monotone independence as an algebraic structure of the monotone Fock space. He defined monotone convolution as the probability distribution of the sum of two monotone independent random variables with an appropriate structure of Harmonic analysis. The reciprocal Cauchy transform is given

$$H_\mu(z) = \frac{1}{G_\mu(z)} \quad \text{-- (1)}$$

where G_μ is the Cauchy transform

$$G_\mu(z) = \int_R \frac{1}{z-x} d\mu(x) \quad \text{---------------------------------- (2)}$$

H_μ is analytic and maps the upper half plane into itself. Also,

$$\inf\ imz > 0\ \frac{imH_\mu(z)}{imz} = 1$$

Consequently, $H_\mu(z)$ is expressed uniquely in the following integral form

$$H_\mu(z) = z + b + \int_R \frac{1+xz}{x-z}\eta(dx) \text{ --------------------- (3)}$$

where $b \in R$ and η is a positive finite measure.

2. COROLLARY

If a $\triangleright$-infinitely divisible distribution v contains an isolated atom at a, v is of the form $v = v(\{a\})\delta_a + v_{ac}$, where v_{ac} is absolutely continuous with respect to the Lebesgue measure and $a \in supp\ v_{ac}$.

$$\{u \in supp\ v\{a\}; \lim_{v \to 0}|G_v(u+iv)| = \infty\} = \emptyset \text{ ------------ (4)}$$

3. LEMMA

Let μ be a probability measure.

(i) $(supp\ \mu)^c \cup (C\backslash R)$ is the maximal domain in which $G_\mu(z)$ is analytic. Similarly, $(supp\ \eta)^c \cup (C\backslash R)$ is the maximal domain in which $H_\mu(z)$ is analytic.

(ii) $\{x \in (supp\ \mu)^c; G_\mu(x) \neq 0\} \subset (supp\ \eta)^c$.

        Similarly, $\{x \in (supp\ \eta)^c; H_\mu(x) \neq 0\} \subset (supp\ \mu)^c$.

        In particular, $a(\eta) \geq a(\mu)$ since $G_\mu(x) \neq 0\ for\ x \in (-\infty, a(\mu))$.

4. THEOREM

The following inequalities hold for probability measures v and μ.

(i)   If $supp\ v \cap (-\infty, 0] \neq \emptyset$ and $supp\ v \cap [0, \infty) \neq \emptyset$, then $a(\mu) \geq a(v \triangleright \mu), b(\mu) \leq b(v \triangleright \mu)$.

(ii)   If $supp\ v \subset (-\infty, 0]$ then $a(\mu) \geq a(v \triangleright \mu)$, $b(v) + b(\mu) \leq b(v \triangleright \mu)$.

(iii)   If $supp\ v \subset [0, \infty)$, then $a(v) + a(\mu) \geq a(v \triangleright \mu), b(\mu) \leq b(v \triangleright \mu)$.

Proof

For a probability measure ρ, let us denote by ρ^x the probability measure $\delta_x \triangleright \rho$. Here $v \triangleright \mu$ is expressed as an expression in its integral form

$$v \triangleright \mu(B) = \int_R \mu^x(B)v(dx) \text{ ----------------------------------- (5)}$$

for Borel sets B as done by N. Muraki. Let $\lambda = v \triangleright \mu$. Then following inequation are satisfied.

$$\begin{cases} a(\rho^x) \geq a(\rho),\ b(\rho^x) \leq b(\rho) + x & \text{for all } x > 0, \\ a(\rho^x) \geq a(\rho) - |x|,\ b(\rho^x) \leq b(\rho) & \text{for all } x < 0, \end{cases} \text{ -------- (6)}$$

Here ρ^x is characterized by $G_{\rho^x} = \frac{G_\rho}{1-xG_\rho}$.

If $x > 0$, then $1 - xG_\rho(z) \neq 0$ for $z \in C\backslash[a(\rho), b(\rho) + x]$ and G_ρ is analytic in this domain. Therefore, the first inequality holds. The second may be proved in a similar way.

Let $J = supp\ \lambda$. In view of the relation $\lambda(A) = \int_R \mu^x(A)dv(x)$, we get $\lambda(J^c) = \int_R \mu^x(J^c)dv(x) = 0$. Hence, we obtain $\mu^x(J^c) = 0$, $v(a), x \in R$. Let us take any x_0 such that $\mu^{x_0}(J^c) = 0$. Then we have $a(\mu^{x_0}) \geq a(\lambda)$ and $b(\mu^{x_0}) \leq b(\lambda)$. If $x_0 > 0$, combining the inequalities $a(\rho^x) \geq a(\rho) - |x|$ and $b(\rho^x) \leq b(\rho)$ for $\rho = \mu^{x_0}$ and $x = -x_0 < 0$, we get

$$a(\mu) = a(\mu^{x_0 - x_0}) \geq a(\lambda) - |x_0|,$$

$$b(\mu) = b(\mu^{x_0 - x_0}) \leq b(\lambda).$$

Similarly if $x_0 < 0$,

$$a(\mu) \geq a(\lambda),$$

$$b(\mu) \leq b(\lambda) + |x_0|.$$

Let us assume the $supp\ v \subset (-\infty, 0]$. Then, we obtain $a(\mu) \geq a(\lambda)$ and $b(\mu) \leq b(\lambda) + |b(v)|$. Since there is a sequence of each $x_0's$ converging to the point $b(v)$. Hence assumption (iii) is proved. The statement (i) and (ii) proved in a similar manner applying the procedure of (2).

5. THEOREM

Let v be a probability measures and let $n \geq 1$ be a natural number.

(i)   If $supp(v^{\triangleright^n}) \subset (-\infty, 0]$ then $supp\ v \subset (-\infty, 0]$ and $|b(v)| \geq \frac{1}{n}|b(v^{\triangleright^n})|$.

(ii)   If $supp(v^{\triangleright^n}) \subset [0, \infty)$ then $supp\ v \subset [0, \infty)$ and $|a(v)| \geq \frac{1}{n}|b(v^{\triangleright^n})|$.

It puts a restriction on the support of a $\triangleright$-infinitely divisible distribution.

Proof

Let $\lambda = v^{\triangleright^n}$.

(i) Let us assume that both $b(v) > 0$ and $b(\lambda) = b(v^{\triangleright^n}) \leq 0$ hold, then there are two possible cases:

(a) $supp\ v \cap [0, \infty) \neq \emptyset$ and $supp\ v \cap (-\infty, 0] \neq \emptyset$

(b) $supp\ v \subset [0, \infty)$

We apply λ and μ with $v^{\triangleright^n}$ and $v^{\triangleright^{n-1}}$ respectively.

In both cases (a) and (b), it holds that $b(v^{\triangleright^{n-1}}) \leq b(\lambda) \leq 0$. Thus we obtain $b(v^{\triangleright^{n-1}}) \leq 0$. We repeat it by induction and obtain $b(v) \leq 0$, which is a contradiction. Therefore,

$b(v) \leq 0$. By the iterative use of (ii), we obtain $b\left(v^{\rhd^n}\right) \geq b(v)$. Hence, the theorem is proved.

6. COROLLARY

The monotone convolution preserves the set $\{\mu; supp\ \mu \subset [0, \infty)\}$ of probability measures.

Proof

If $supp\ \mu \subset [0, \infty)$ and $supp\ v \subset [0, \infty)$, $H_{\mu \rhd v} = H_\mu \circ H_v$ is analytic in $C \backslash [0, \infty)$. Since $H_{\mu \rhd v}(-0) = H_\mu \circ H_v(-0) \leq H_\mu(-0) \leq 0$. We obtain $supp\ (\mu \rhd v) \subset [0, \infty)$. This property also holds for Boolean convolution. We apply here the operator-theoretic realization of monotone independent random variables. Let $mn(\mu) = \int_R x^n \mu(dx)$ be the n-th moment of a probability measure μ.

7. COROLLARY

Let μ be the probability measure and let $n \geq 1$ be a natural number. Then the following conditions are equivalent.

(i)　　　$m_{2n}(\mu) < \infty$,

(ii)　　H_μ has the expression $H_\mu(z) = z + a + \int_R \frac{\rho(dx)}{x-z}$, where $a \in R$ and ρ is a positive finite measure satisfying $m_{2n-2}(\rho) < \infty$,

(iii)　　there exists $a_1, a_2, \ldots\ldots\ldots\ldots\ldots a_n \in R$ such that
$$H_\mu(z) = z + a_1 + \frac{a_2}{z} + \ldots\ldots\ldots + \frac{a_{2n}}{z^{2n-1}} + 0\left(|z|^{-(2n-1)}\right) \text{ ---(7)}$$
for $z = iy(y \to \infty)$.

If (iii) holds, for any $\delta > 0$ the expansion (7) holds for $z \to \infty$ satisfying $Im z > \delta |Re\ z|$. Moreover, we have $a_{k+2} = -m_k(\rho)$, $0 \leq k \leq 2n - 2$.

8. THEOREM

Let μ and v be the probability measure and let $n \geq 1$ be a natural number. If $m_{2n}(\mu) < \infty$ and $m_{2n}(v) < \infty$, then $m_{2n}(\mu \rhd v) < \infty$. Moreover, we have

$$m_1(\mu \rhd v) = m_1(\mu) + m_1(v) + \sum_{k=1}^{i-1} \sum_{\substack{j_0+j_1+\cdots\ldots+j_k=i-k \\ 0 \leq j_p,\ \ 0 \leq p \leq k}} m_k(\mu) m_{j_0}(v) \ldots\ldots\ldots m_{j_k}(v)$$

-------------------- (8)

for $1 \leq k \leq 2n$

Proof

We find that $Im H_v(z) \geq Im z$. For any $\delta > 0$, there exists $M = M(\delta) > 0$ such that

$$Im H_v(iy) \geq y > \delta |Re H_v(iy)| \text{ for } y > M \text{ ----------------- (9)}$$

By suitable application of condition (7), we obtain

$$H_\mu\big(H_v(iy)\big) = H_v(iy) + a_1 + a_2 G_v(iy) + \cdots + a_{2n} G_v(iy)^{2n-1} + R\big(H_v(iy)\big)$$

where $z^{2n-1} R(z) = \int_R \frac{x^{2n-1}}{x-z} \rho(dx) \to 0$ as $z \to \infty$ satisfying $Im z > \delta |Re\, z|$ for a fixed $\delta > 0$. We thus obtain

$$y^{2n-1}\big|R\big(H_v(iy)\big)\big| \leq |H_v(iy)|^{2n-1}\big|R\big(H_v(iy)\big)\big| \to 0 \quad \text{as } y \to \infty$$

taking help of the condition (9). Thus $R\big(H_v(iy)\big) = 0\big(y^{-(2n-1)}\big)$. Expanding $H_v(z)$ in the form (7), we find that there exists $0_1, 0_2, \ldots \ldots \ldots, 0_{2n} \in R$ such that $H_\mu(H_v(z)) = z + 0_1 + 0_2 z + \cdots \ldots \ldots + \frac{0_{2n}}{z^{2n-1}} + 0\big(|z|^{-(2n-1)}\big)$ for $z = iy(y \to \infty)$. Hence the theorem is proved.

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