Convolution Structure of Two Dimensional Offset Fractional Fourier Transform

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Abstract : The Fractional Fourier transform (FRFT) is generalization of Fourier transform. The Fractional Fourier transform has been investigated by many different authors and they proved it is very useful in solving some problem in quantum physics, optics, filter design, image processing, pattern recognition, wavelet transform. Fractional Fourier transform performs a rotation of signal in the time-frequency plane and it has many applications in time varying signal analysis, pattern recognition algorithms makes extensive use of convolution because of its translation invariance property. The Offset Fractional Fourier transform is the space shifted frequency modulated version of original one. In this paper we discussed the Two Dimensional Offset Fractional Fourier transform in distributional generalize sense. Convolution theorem for the Two Dimensional Offset Fractional Fourier transform is also proved.

***Index Terms* - Fourier transform, Fractional Fourier Transform, Two-dimensional Offset Fractional Fourier transform, Generalized Function, Testing function Space.**

I. INTRODUCTION

The Fractional Fourier transform (FRFT) is the generalization of classical Fourier transform. It can analyze the signal in between the time and frequency domain. Fractional Fourier transform has the effect of rotating the space-frequency representation of a signal; its domain corresponds to oblique axes in the space-frequency plane (phase space). FRFT has received much attention in recent years. Several applications if FRFT have been suggested. In particular, many signal and image processing applications have been developed on the basis of the FRFT [2, 3, 4]. Several two-dimensional optical implementations have been discussed previously [5, 6]. Thus, Fractional Fourier transform is very useful tool for signal processing and has many applications such as optical filter design, signal synthesis, solving differential equations, phase retrieval, and pattern recognition, quantum mechanics, fractional convolution and correlation, beam forming etc. Fractional Fourier transform has been further generalized into Linear Canonical transform (LCT) [7]. The application of LCT are similar to those of the Fractional Fourier transform, it is especially useful for optical system analysis and synthesis.

Offset Fractional Fourier transform is useful in optics. They are especially useful for analyzing optical system with prisms or shifted lenses [8]. . The Offset FT, FRFT and LCT have the more parameters than the original FT, FRFT and LCT respectively. More analysis work in optics can be done by these transforms. We can use their Eigenfunction to analyze the self-imaging phenomena of optical system. Their Eigen functions are also useful for resonance phenomena analysis, derivation of the Eigenvectors of Offset DFT, fractal theory development, mode selection and phase retrieval.

Convolution is a most important tool in pattern recognition because of its translation invariance property. Also convolution is a powerful way of characterizing the input output relationship of time invariance system. By using convolution theorem we can tabulate atomic scattering factors by working out the diffraction pattern of atom place at origin. It is useful for extraction of information in communication engineering [2].

Two-Dimensional Offset Fractional Fourier Transform $\left[F_\alpha^{\tau,\eta,\zeta,\gamma} f(t,x)\right](s,u)$ of function $f(t,x)$ through an angle α is defined as

$$\left[F_\alpha^{\tau,\eta,\zeta,\gamma} f(t,x)\right](s,\mu) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(t,x)k_\alpha(t,s-\eta,x,\mu-\gamma)\,dt\,dx$$

where

$$K_\alpha(t,s-\eta,x,\mu-\gamma) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}\,e^{i(s\tau+\mu\zeta)}.\,e^{\frac{i}{2\sin\alpha}.\{[t^2+(s-\eta)^2+x^2+(\mu-\gamma)^2]\cos\alpha-2[(s-\eta)t+(\mu-\gamma)x]\}}$$

Let $a(\alpha) = \frac{\cot\alpha}{2}$, $p(\alpha) = \sec\alpha$, $c(\alpha) = \sqrt{\frac{1-i\cot\alpha}{2\pi}}$

Therefore $K_\alpha(t,s-\eta,x,\mu-\gamma) = c.\,e^{i(s\tau+\mu\zeta)}.\,e^{ia.\{[t^2+(s-\eta)^2+x^2+(\mu-\gamma)^2]-2p[(s-\eta)t+(\mu-\gamma)x]\}}$

Zayed had defined fractional Fourier type convolution for as one dimensional as in [14], similarly we defined for any function $f(t,x)$.

$$\tilde{f}(t,x) = f(t,x).\,e^{ia(t^2+x^2)} \qquad \ldots\ldots\ldots(1.1)$$

Then for any two functions f and g the operation $*$ is defined as

$$h(t,x) = (f*g)(t,x) = c.\,e^{i(s\tau+\mu\zeta)}e^{-ia(t^2+x^2)}(\tilde{f}*\tilde{g})(t,x) \qquad \dots\dots\dots(1.2)$$

Convolution of two-dimensional offset Fractional Fourier transform is defined as

$$h(t,x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v,w)g(t-\gamma, x-w)\,dv\,dw$$

$$h(t,x) = (f*g)(t,x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v,w)g(t-\gamma, x-w)\,dv\,dw \quad] \qquad \dots\dots\dots(1.3)$$

In the present paper we proposed convolution structure for Two-Dimensional Offset Fractional Fourier transform.

II. Testing function space E:

An infinitely deferential complex valued smooth function on $\phi(R^n)$ belongs to (R^n), if for each compact $I \subset S_{a,b}$ where

$$S_{a,b} = \{t; x: t\ x \in R^n,\ |t| \le a, |x| \le b, a < 0, b < 0\}, \qquad I \in R^n$$

$$\gamma_{I,l,q}(\phi) = \sup_{t,x,\in I}\left|D_{t,x}^{l,q}\phi(t,x)\right| < \infty, \qquad l,q = 0,1,2\dots\dots\dots \qquad \dots\dots\dots(2.1)$$

III. Distributional Two-Dimensional offset Fractional Fourier Transform

The Two-Dimensional Offset Fractional Fourier Transform $[F_\alpha^{\tau,\eta,\zeta,\gamma}f(t,x)](s,u)$of generalization function $f(t,x)$ through an angle α is defined as,

$$[F_\alpha^{\tau,\eta,\zeta,\gamma}f(t,x)](s,u) = \langle f(t,x)\ K_\alpha(t, s-\eta, x, u-\gamma)\rangle$$

where

$$K_\alpha(t,s-\eta,x,u-\gamma) = C_{1\alpha}e^{i(s\tau+u\zeta)}e^{iC_{2\alpha}[((s-\eta)^2+t^2+(u-\gamma)^2+x^2)\cos\alpha - 2((s-\eta)t+(u-\gamma)x)]}$$

where

$$C_{1\alpha} = \sqrt{\frac{1-i\cot\alpha}{2\pi}} \quad \text{and} \quad C_{2\alpha} = \frac{1}{2\sin\alpha} \qquad \dots\dots\dots(3.1)$$

IV. Convolution Theorem of two dimensional offset Fractional Fourier transform

Let $h(t,x) = (f*g)(t,x)$ and $F_\alpha^{\tau,\eta,\zeta,\gamma}, G_\alpha^{\tau,\eta,\zeta,\gamma}, H_\alpha^{\tau,\eta,\zeta,\gamma}$ denotes the two-dimensional offset FRFT of f, g, h respectively then

$$H_\alpha^{\tau,\eta,\zeta,\gamma}(s,u) = F_\alpha^{\tau,\eta,\zeta,\gamma}(s,u).\,G_\alpha^{\tau,\eta,\zeta,\gamma}(s,u).\,e^{-ia((s-\eta)^2+(\mu-\gamma)^2)}$$

Proof:

From the definition of two dimensional offset Fractional Fourier Transform

$$H_\alpha^{\tau,\eta,\zeta,\gamma}(s,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(t,x)K_\alpha(t,s-\eta,x,\mu-\gamma)\,dt\,dx$$

$$= c.\,e^{i(s\tau+\mu\zeta)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(f*g)(t,x)\; e^{ia.\{[t^2+(s-\eta)^2+x^2+(\mu-\gamma)^2]-2pt[(s-\eta)t+(\mu-\gamma)x]\}}dt\,dx$$

$$= c.\,e^{i(s\tau+\mu\zeta)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} c.\,e^{i(s\tau+\mu\zeta)}e^{-ia(t^2+x^2)}(\tilde{f}*\tilde{g})(t,x).\; e^{ia.\{[t^2+(s-\eta)^2+x^2+(\mu-\gamma)^2]-2pt[(s-\eta)t+(\mu-\gamma)x]\}}dt\,dx$$

$$= c^2.\left[e^{i(s\tau+\mu\zeta)}\right]^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-ia(t^2+x^2)}\tilde{f}(v,w).\,\tilde{g}(t-\gamma,x-w).\; e^{ia.\{[t^2+(s-\eta)^2+x^2+(\mu-\gamma)^2]-2pt[(s-\eta)t+(\mu-\gamma)x]\}}dt\,dx\,dv\,dw$$

$$= c^2.\left[e^{i(s\tau+\mu\zeta)}\right]^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v,w).\,g(t-\gamma,x-w)\; e^{i[2a\gamma^2+2aw^2-2atv-2axw+at]}\,e^{i[ax^2+a(s-\eta)^2-a(\mu-\gamma)^2-2apt(s-\eta)-2apx(u-\gamma)]}dt\,dx\,dv\,dw$$

Put

$$t = m + \gamma \quad \text{and} \quad x = n + w$$
$$t - \gamma = m \quad \text{and} \quad x - w = n$$
$$dt = dm \quad \text{and} \quad dx = dn$$

$$H_{\alpha}^{\tau,\eta,\zeta,\gamma}(s,u) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v,w).\,g(m,n).\,e^{i[2a\gamma^2+2aw^2-2a\gamma(m+\gamma)-2aw(n+w)+a(m+v)]}$$
$$e^{i[a(n+w)^2+a(s-\eta)^2+a(\mu-\gamma)^2-2ap(m+\gamma)(s-\eta)-2ap(\mu-\gamma)(n+\mu)]}dm\,dn\,d\gamma d$$

$$= c^2.\,e^{i(s\tau+\mu\zeta)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v,w).\,g(m,n).\,e^{ia\left[\gamma^2+(s-\eta)^2+w^2+(\mu-\gamma)^2-2p[(s-\eta)\gamma+(u-\gamma)w]\right]}$$
$$e^{ia\left[m^2+n^2-2p[(s-\eta)m+(u-\gamma)n]\right]}dm\,dn\,d\gamma dw$$

$$= e^{ia(s-\eta)^2}e^{ia(\mu-\gamma)^2}\left[c.\,e^{i(s\tau+\mu\zeta)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v,w)\,e^{ia.\left[\gamma^2+(s-\eta)^2+w^2+(\mu-\gamma)^2-2p[(s-\eta)\gamma+(u-\gamma)w]\right]}\,d\gamma\,dw\right]$$
$$\left[c.\,e^{i(s\tau+\mu\zeta)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(m,n)\,e^{ia\left[m^2+(s-\eta)^2-n^2+(\mu-\gamma)^2-2p[(s-\eta)m+(u-\gamma)n]\right]}\,dm\,dn\right]$$

$$= e^{-ia[\,(s-\eta)^2+(\mu-\gamma)^2]}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\gamma,w)\,k_{\alpha}(\gamma,s-\eta,w,\mu-\gamma)\,d\gamma\,dw\right].$$
$$\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(m,n)\,k_{\alpha}(m,s-\eta,\eta,\mu-\gamma)\,d\gamma\,dw\right]$$

$$= e^{-ia[\,(s-\eta)^2+(\mu-\gamma)^2]}\,.\,F_{\alpha}^{\tau,\eta,\zeta,\gamma}(s,\mu).\,G_{\alpha}^{\tau,\eta,\zeta,\gamma}(s,\mu)$$

Therefore

$$H_{\alpha}^{\tau,\eta,\zeta,\gamma}(s,u) = F_{\alpha}^{\tau,\eta,\zeta,\gamma}(s,u).\,G_{\alpha}^{\tau,\eta,\zeta,\gamma}(s,u).\,e^{-ia((s-\eta)^2+(\mu-\gamma)^2)}$$

V. CONCLUSION
VI.

In the present work Convolution structure for Two-Dimensional Offset Fractional Fourier transform is designed. Convolution structure for Two-Dimensional Offset Fractional Fourier transform may be applicable in computer version, image processing, signal processing, statistics and probability.

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