ANALYSIS OF M^X/G/1 QUEUE WITH STATE DEPENDENT ARRIVAL, FEEDBACK AND VACATION

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Abstract: This paper investigates on batch arrival queueing system with feedback and vacation. If the server is idle upon the arrival, then one of the customers in the arriving batch receives service immediately and the rest joins the queue. On the other hand, if the server is found to be busy or on vacation, then all the customers in the batch joins the queue and waits for service until the busy server becomes free. If the customer is satisfied with the service provided, then he may depart the system otherwise he is allowed to join the tail of the queue as a feedback customer. After completion of service, the server takes a compulsory vacation of random length. The Steady state distributions of the server state are deduced and the mean number of customers in the system and mean number of customers in the queue are obtained.

Keywords: Batch arrival, state dependent, feedback and compulsory vacation.

I INTRODUCTION

The service rendered by a single server queuing system has been founds to be useful to analyse many practical situations arising in packet transmissions of communication networks, multimedia communication, central processors etc., and the first use of the term "queuing system" occurred in 1951 in the Journal of the Royal Statistical Society, when D.C.Kendall [1] published his article on "Some Problems in the Theory of Queues" A comprehensive review of vacation queuing models, method and applications can e found in the survey of Doshi [2] and the text book of Tian and Zhang [3]. A short survey by Jain and Upadhyaya [4] analyzed the recent developments in the field. For an M/G/1 queue with Bernoulli feedback. Thangaraj and Vanitha [8], Choi and Tae-Sung [5], Borthakur and Choudhurya [6] derived on a batch arrival Poisson queue with generalized vacation. Saravanarajan and Chandrasekaran [7] analyzed on batch arrival queueing system with state dependent arrival, feedback and vacation. It is quite evident that batch arrival queuing systems have real life application. As we finds there is no model that combines batch arrival queue with state dependent analysis, feedback and vacation.

This paper is organized as follows. In Section II, we give a brief description of the mathematical model. Section III with the governing equations of the model. Derivation of the Steady state probability generating function has been done in Section V. Some performance measures of this model are derived in Section VI. Finally in Section VII, mean queue size and mean system size is presented.

II MATHEMATICAL DESCRIPTION OF THE MODEL

Arrival: Assume that batch of customers arrive at the system according to a compound Poisson process with state dependent rates λ_1, λ_2,λ_3(i.e., in arrival state, busy state, vacation state). At every arrival epoch, a batch of k customers arrive at the system with probability C_k, k=1,2,… Let $C(z) = \sum_{k=1}^{\infty} C_k z^k$, be the generating function of the batch size distribution with first two moments m_1 and m_2. The server will provide service on FCFS discipline.

Service: The server provides service to the customers one by one in the batch. If the arriving batch of customers finds the server is free, one of the arrivals receives service immediately and the rest joins the queue. Otherwise, all the arriving batch of customers joins the queue. The service time follows a general distribution with distribution function B(x), density function b(x), Laplace Stieltje's transform $B^*(\bullet)$, n^{th} factorial moments μ_n and conditional completion rate $\mu(x)$ dx be the conditional probability of completion of service in the interval (x, x+dx], given that the elapsed time is x, so that

$$\mu(x) = \frac{b(x)}{1 - b(x)}$$

and therefore

$$b(r) = \mu(r) e^{-\int_0^\infty \mu(x)dx}$$

Feedback: After completing the service, if the customer is unsatisfied with his service then he can immediately join the queue as a feedback customer for receiving another service with probability q or departs the system with its complementary probability 1-q. Here, the feedback customer will be served in the order in which he joined the tail of the queue.

Vacation: As soon as the completion of service of a customer, the server will take a compulsory vacation of random length. The vacation time follows a general distribution with distribution function V(x), density function v(x), Laplace Stieltje's transform $V^*(\bullet)$, n^{th} factorial moments μ_n and conditional completion rate $\beta(x)$ dx be the conditional probability of completion of service in the interval (x, x+dx], given that the elapsed time is x, so that

$$\beta(x) = \frac{v(x)}{1 - v(x)}$$

and therefore

$$v(r) = \beta(r) e^{-\int_0^\infty \beta(x)dx}$$

On returning from vacation the server instantly starts serving the customer at the head of the queue, if any.

Various stochastic processes involved in the system are assumed to be independent of each other.

III DEFINITIONS AND GOVERNING EQUATIONS OF THE SYSTEM

We define

$P_n(t)$ is the probability that time t, there are $n \geq 0$ customers in the queue excluding the one being provided the elapsed service time for this customer is x. Accordingly, $P_n(t) = \int_0^\infty P_n(x,t)dx$ is the probability that at time t, there are n customers in the queue excluding the service irrespective of the value of x.

$V_n(x,t)$is the probability that time t, there are $n \geq 0$ customers in the queue and the server is under vacation with elapsed vacation time x.Accordingly, $V_n(t) = \int_0^\infty V_n(x,t)dx$ is the probability that at time t, there are n customers in the queue and the server is under vacation irrespective of the value of x.

$I(t)$is the probability that at time t, there are no customers in the system and the server is idle but available in the system.

IV STEADY STATE EQUATION

The Steady state equation that governs the model under consideration using supplementary variable are given below

$$\lambda_1 I = \int_0^\infty V_0(x,t)\beta(x)dx \tag{4.1}$$

$$\frac{d}{dx} P_n(x) = -(\lambda_2 + \mu(x))P_n(x) + \lambda_2 \sum_{k=1}^{n} C_k P_{n-k}(x), \quad n \geq 1 \tag{4.2}$$

$\frac{d}{dx}V_n(x) = -(\lambda_3 + \beta(x))V_n(x) + \lambda_3 \sum_{k=1}^{n} C_k V_{n-k}(x)$, n ≥ 1
(4.3)
With boundary conditions
$P_0(0) = I\lambda_1 C_1 + \int_0^\infty V_1(x)\beta(x)dx$
(4.4)
$P_n(0) = I\lambda_1 C_{n+1} + \int_0^\infty V_{n+1}(x)\beta(x)dx$, n ≥ 1
(4.5)
$V_0(0) = (1-q)\int_0^\infty P_0(x)\mu(x)dx$
(4.6)
$V_n(0) = (1-q)\int_0^\infty P_n(x)\mu(x)dx + q\int_0^\infty P_{n-1}(x)\mu(x)dx,$ n ≥ 1
(4.7)

Define the probability generating functions:
$P(x,z) = \sum_{n=0}^{\infty} P_n(x)z^n$ and $V(x,z) = \sum_{n=0}^{\infty} V_n(x)z^n$

The normalizing condition
$\mathrm{I} + \mathrm{P}(1) + \mathrm{V}(1) = 1$
(4.8)

V STEADY STATE PROBABILITY GENERATING FUNCTION

Multiplying equations (4.2) to (4.7) by z^n and summing over all possible values of n, we get the following equation
$\left[\frac{d}{dx} + \lambda_2 - \lambda_2 C(z) + \mu(x)\right] P(x,z) = 0$
(5.1)
$\left[\frac{d}{dx} + \lambda_3 - \lambda_3 C(z) + \beta(x)\right] V(x,z) = 0$
(5.2)
$P(0,z) = \frac{\lambda_1 I}{z}(C(z) - 1) + \frac{1}{z}\int_0^\infty V(x,z)\ \beta(x)dx$
(5.3)
$V(0,z) = (1-q+qz)\ \int_0^\infty P(x,z)\ \mu(x)dx$
(5.4)
Solving the partial differential equations (5.1), we get
$\frac{d}{dz}\sum_{n=0}^{\infty} P_n(x)z^n = -\sum_{n=0}^{\infty}(\lambda_2 + \mu(x))P_n(x)z^n + \sum_{n=0}^{\infty}\lambda_2 \sum_{k=1}^{n} C_k P_{n-k}(x)z^{n-k+k}$
$\frac{d}{dz}\ P(x,z) = -(\lambda_2 + \mu(x))P(x,z) + \lambda_2 C(z)P(x,z)$
$\left[\frac{d}{dz} + \lambda_2 - \lambda_2 C(z) + \mu(x)\right]\ P(x,z) = 0$

Solution of the above equation is given by
$P(x,z) = \ P(0,z)e^{-(\lambda_2 - \lambda_2 c(z))x}[1 - B(x)]$
(5.5)

Solving the partial differential equations (5.4), we get
$\frac{d}{dz}\sum_{n=0}^{\infty} V_n(x)z^n = -\sum_{n=0}^{\infty}(\lambda_3 + \beta(x))V_n(x)z^n + \sum_{n=0}^{\infty}\lambda_3 \sum_{k=1}^{n} C_k V_{n-k}(x)z^{n-k+k}$
$\frac{d}{dz}\ V(x,z) = -(\lambda_3 + \beta(x))V(x,z) + \lambda_3 C(z)V(x,z)$
$\left[\frac{d}{dz} + \lambda_3 - \lambda_3 C(z) + \beta(x)\right]\ V(x,z) = 0$

Solution of the above equation is given by

$$V(x,z) = V(0,z)e^{-(\lambda_2-\lambda_2 C(z))x}[1-V(x)]$$
(5.6)

Using equation (5.4) in equation (5.5) and on solving we get

$$V(0,z) = (1-q+qz)P(0,z)B^*(\lambda_2-\lambda_2 C(z))$$
(5.7)

Using equations (5.6) and (5.7) in equation (5.3) and on simplifying we get

$$P(0,z) = \frac{I\lambda_1(C(z)-1)}{z-(1-q+qz)B^*(\lambda_2-\lambda_2 C(z))V^*(\lambda_3-\lambda_3 C(z))}$$
(5.8)

Substituting equation (5.8) in (5.7), gives

$$V(0,z) = \frac{I\lambda_1(C(z)-1)(1-q+qz)B^*(\lambda_2-\lambda_2 C(z))}{z-(1-q+qz)B^*(\lambda_2-\lambda_2 C(z))V^*(\lambda_3-\lambda_3 C(z))}$$
(5.9)

Applying the expressions of $P(0,z)$ and $V(0,z)$ in equations (5.5) and (5.6), we obtain

$$P(x,z) = \frac{I\lambda_1(C(z)-1)e^{-(\lambda_2-\lambda_2 C(z))x}[1-B(x)]}{z-(1-q+qz)B^*(\lambda_2-\lambda_2 C(z))V^*(\lambda_3-\lambda_3 C(z))}$$
(5.10)

$$V(x,z) = \frac{I\lambda_1(C(z)-1)(1-q+qz)B^*(\lambda_2-\lambda_2 C(z))e^{-(\lambda_3-\lambda_3 C(z))}[1-V(x)]}{z-(1-q+qz)B^*(\lambda_2-\lambda_2 C(z))V^*(\lambda_3-\lambda_3 C(z))}$$
(5.11)

VI PERFORMANCE MEASURES

Define the partial generating function $\psi(z) = \int_0^\infty \psi(z,x)\,dx$ for any generating function $\psi(z,x)$.

The probability generating function of the queue size when server is busy is given by

$$P(z) = \int_0^\infty P(x,z)dx = \frac{{\lambda_1}/{\lambda_2}\, I(B^*(\lambda_2-\lambda_2 C(z))-1)}{z-(1-q+qz)B^*(\lambda_2-\lambda_2 C(z))V^*(\lambda_3-\lambda_3 C(z))}$$

Then,

$$P(1) = \lim_{z\to 1} P(z)$$

$$P(1) = \frac{I\lambda_1 m_1\mu_1}{1-q-m_1(\lambda_2\mu_1+\lambda_3\beta_1)}$$
(6.1)

The probability generating function of the queue size when the server is on vacation is given by

$$V(z) = \int_0^\infty V(x,z)dx = \frac{({\lambda_1}/{\lambda_3})IB^*(\lambda_2-\lambda_2 C(z))(V^*(\lambda_3-\lambda_3 C(z))-1)(1-q+qz)}{[z-(1-q+qz)B^*(\lambda_2-\lambda_2 C(z))V^*(\lambda_3-\lambda_3 C(z))]}$$

Then,

$$V(1) = \lim_{z\to 1} V(z)$$

$$V(1) = \frac{I\lambda_1 m_1\beta_1}{1-q-m_1(\lambda_2\mu_1+\lambda_3\beta_1)}$$
(6.2)

Using equation (6.1) and (6.2) and solving for I we get

$$I = \frac{1-q-m_1(\lambda_2\mu_1+\lambda_3\beta_1)}{(1-q)+m_1\mu_1(\lambda_1-\lambda_2)+m_1\beta_1(\lambda_1-\lambda_3)}$$
(6.3)

Now using equation (6.3), equation (6.1) and (6.2) yield the following results

The steady state probability that the server is busy is

$$P = \frac{\lambda m_1 \mu_1}{(1-q)+m_1\mu_1(\lambda_1-\lambda_2)+m_1\beta_1(\lambda_1-\lambda_3)}$$

(6.4)

The steady state probability that the server on vacation is

$$V = \frac{\lambda m_1 \beta_1}{(1-q)+m_1\mu_1(\lambda_1-\lambda_2)+m_1\beta_1(\lambda_1-\lambda_3)}$$

(6.5)

VII MEAN QUEUE SIZE AND SYSTEM SIZE

The probability generating function for the number of customers in the queue is

$$P_q(z) = P(z) + V(z)$$

$$= \frac{I\left[\left(\frac{\lambda_1}{\lambda_2}\right)B^*(h_3(z)-1)+\left(\frac{\lambda_1}{\lambda_3}\right)(1-q-qz)B^*\left(h_3(z)\right)(V^*\left(h_4(z)\right)-1\right]}{D(z)}$$

(7.1)

where,

$$h_3(z) = \lambda_2 - \lambda_2 C(z)$$

$$h_4(z) = \lambda_3 - \lambda_3 C(z)$$

$$D(z) = z - (1-q-qz)B^*\left(h_3(z)\right)V^*(h_4(z))$$

The mean number of customers in the queue is

$$L_q = \lim_{z\to 1}\frac{d}{dz}P_q(z)$$

$$= \frac{D'(1)N_r''(1)-N_r'(1)D''(1)}{2(D'(1))^2}$$

(7.2)

where, $N_r'(z) and D(z)$ denote the numerator and denominator of $P_q(z)$

$$N_r'(1) = I\lambda_1 m_1(\mu_1+\beta_1)$$

$$N_r''(1) = I\lambda_1\{[\lambda_2 m_1^2\mu_2 + m_2\mu_1] + [2m_1\beta_1 q + 2\lambda_2 m_1^2\mu_1\beta_1 + m_2\beta_1 + \lambda_3 m_1^2\beta_2]\}$$

The probability generating function for the number of customers in the system is

$$P_s(z) = I + z(P_q(z)$$

$$= \frac{I\left[z+(1-q-qz)B^*(h_3(z)(\frac{\lambda_1}{\lambda_3}z(V^*(h_4(z))-1)-\left(V^*\left(h_4(z)\right)\right)+\left(\frac{\lambda_1}{\lambda_3}\right)z(B^*(h_3(z)-1)\right]}{D(z)}$$

(7.3)

The mean number of customers in the system is

$$L_s = \lim_{z\to 1}\frac{d}{dz}P_s(z)$$

$$= L_q + \mathrm{I}$$

(7.4)

Conclusion

The steady state analysis of batch arrival of queueing system with state dependent arrival, feedback and vacation is considered. Assumed that the service time and vacation time follows a general distribution, the joint distribution of the state of the server and the number of customers in the system and in the queue is derived in the steady state. The explicit expressions of some performance measure are given.

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