

# Integral Transform of p-k Mittag-Leffler Function

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**Abstract:** In this paper, we evaluate Mellin-Barnes integral representation of p-k Mittag-Leffler Function with their several special cases. The relationship of p-k Mittag-Leffler Function with Fox H-Function and Wright hypergeometric function is also establish. In last we obtained its Euler transform, Laplace Transform, Mellin transform and several special cases.

**MSC(2011):** 33E12, 44Axx

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## 1 Introduction

The two parameter pochhammer symbol is recently introduce by [6](equation 2.1), in the form,

### 1.1 Definition

Let  $x \in C; k, p \in R^+ - \{0\}$  and  $Re(x) > 0, n \in N$ , the p-k Pochhammer Symbol (i.e. Two Parameter Pochhammer Symbol),  ${}_p(x)_{n,k}$  is given by

$${}_p(x)_{n,k} = \left(\frac{xp}{k}\right)\left(\frac{xp}{k} + p\right)\left(\frac{xp}{k} + 2p\right)\dots\dots\left(\frac{xp}{k} + (n-1)p\right). \quad (1)$$

And the Two Parameter Gamma Function is given by [6](equation 2.6, 2.7 and 2.14),

### 1.2 Definition

For  $x \in C/kZ^-; k, p \in R^+ - \{0\}$  and  $Re(x) > 0, n \in N$ , the p - k Gamma Function (i.e. Two Parameter Gamma Function),  ${}_p\Gamma_k(x)$  as,

$${}_p\Gamma_k(x) = \frac{1}{k} \lim_{n \rightarrow \infty} \frac{n! p^{n+1} (np)^{\frac{x}{k}}}{{}_p(x)_{n+1,k}}. \quad (2)$$

or

$${}_p\Gamma_k(x) = \frac{1}{k} \lim_{n \rightarrow \infty} \frac{n! p^{n+1} (np)^{\frac{x-1}{k}}}{{}_p(x)_{n,k}}. \quad (3)$$

The integral representation of p - k Gamma Function is given by,

$${}_p\Gamma_k(x) = \int_0^\infty e^{-\frac{t^k}{p}} t^{x-1} dt. \quad (4)$$

The Beta function,

$$B(m,n) = \frac{\Gamma(n)\Gamma(m)}{\Gamma(m+n)} = \int_0^1 t^{n-1} (1-t)^{m-1} dt; (Re(n) > 0, Re(m) > 0) \quad (5)$$

Let  $k, p \in R^+ - \{0\}; \alpha, \beta, \gamma \in C/kZ^-; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0$  and  $q \in (0,1) \cup N$ .

The p - k Mittag-Leffler function denoted by  ${}_p E_{k,\alpha,\beta}^{\gamma,q}(z)$  and defined as,

$${}_p E_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p\Gamma_k(n\alpha + \beta)} \frac{z^n}{n!}. \quad (6)$$

Where  ${}_p(\gamma)_{nq,k}$  is two parameter Pochhammer symbol given by equation (1.1) and  ${}_p\Gamma_k(x)$  is the two parameter Gamma function given by equation (1.3).

**Particular cases :** For some particular values of the parameters  $p, q, k, \alpha, \beta, \gamma$  we can obtain certain Mittag-Leffler functions, defined earlier:

(a) For  $q = 1$  equation (2.1), reduces in generalized form of k- Mittag-Leffler functions defined as,

$${}_p E_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k} z^n}{{}_p\Gamma_k(n\alpha + \beta)(n!)} . \quad (7)$$

(b) For  $p = k$  equation (2.1), reduces in Generalized k- Mittag-Leffler functions defined by [5],

$${}_k E_{k,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{{}_k(\gamma)_{nq,k} z^n}{{}_k\Gamma_k(n\alpha + \beta)(n!)} = G E_{k,\alpha,\beta}^{\gamma,q}(z) . \quad (8)$$

(c) For  $p = k, q = 1$  equation (2.1), reduces in k - Mittag-Leffler functions defined by [3],

$${}_k E_{k,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{nq,k} z^n}{{}_k\Gamma_k(n\alpha + \beta)(n!)} = E_{k,\alpha,\beta}^{\gamma}(z) . \quad (9)$$

(d) For  $p = k$  and  $k = 1$  equation (2.1), reduces in Mittag-Leffler functions defined by [1],

$${}_1 E_{1,\alpha,\beta}^{\gamma,q}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(n\alpha + \beta)(n!)} = E_{\alpha,\beta}^{\gamma,q}(z) . \quad (10)$$

(e) For  $p = k, q = 1$  and  $k = 1$  equation (2.1), reduces in Mittag-Leffler functions defined by [8],

$${}_1 E_{1,\alpha,\beta}^{\gamma,1}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n z^n}{\Gamma(n\alpha + \beta)(n!)} = E_{\alpha,\beta}^{\gamma}(z) . \quad (11)$$

(f) For  $p = k, q = 1, k = 1$  and  $\gamma = 1$  equation (2.1), reduces in Mittag-Leffler functions defined by [3],

$${}_1 E_{1,\alpha,\beta}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)} = E_{\alpha,\beta}(z) . \quad (12)$$

(g) For  $p = k, q = 1, k = 1, \gamma = 1$  and  $\beta = 1$  equation (2.1), reduces in Mittag-Leffler functions defined by [4],

$${}_1 E_{1,\alpha,1}^{1,1}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)} = E_{\alpha}(z) . \quad (13)$$

Relation between classical Pochhammer symbol and two parameter Pochhammer symbol are given below [6].

**Proposition 1.** Let  $x \in C \setminus kZ^-$ ,  $k, p, r, s \in R^+ - \{0\}$  and  $Re(x) > 0$ ,  $n \in N$ , then the following identity holds,

$${}_r\Gamma_s(x) = \frac{k}{s} \left( \frac{r}{p} \right)^{\frac{x}{s}} {}_p\Gamma_k \left( \frac{kx}{s} \right) . \quad (14)$$

and particular case,

$${}_r\Gamma_k(x) = \left( \frac{r}{p} \right)^{\frac{x}{k}} {}_p\Gamma_k(x) . \quad (15)$$

**Proposition 2.** Let  $x \in C \setminus kZ^-$ ,  $k, p \in R^+ - \{0\}$  and  $Re(x) > 0$ ,  $n, q \in N$ , then the following identity holds,

$${}_p(x)_{nq,k} = \left( \frac{p}{k} \right)^{nq} (x)_{nq,k} . \quad (16)$$

and particular case,

$${}_p(x)_{nq,k} = (p)^{nq} \left( \frac{x}{k} \right)_{nq} . \quad (17)$$

Wright generalized hypergeometric function [10],

$${}_p\psi_q \left[ \begin{array}{c} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{array} z \right] = \sum_{n=0}^{\infty} \frac{\prod_{i=1}^p \Gamma(\alpha_i + A_i n) z^n}{\prod_{j=1}^q \Gamma(\beta_j + B_j n) n!}. \quad (18)$$

$${}_p\psi_q \left[ \begin{array}{c} (\alpha_1, A_1), \dots, (\alpha_p, A_p); \\ (\beta_1, B_1), \dots, (\beta_q, B_q); \end{array} z \right] = H_{p,q+1}^{1,p} \left[ \begin{array}{c} (1-\alpha_1, A_1), \dots, (1-\alpha_p, A_p); \\ (0,1), (1-\beta_1, B_1), \dots, (1-\beta_q, B_q); \end{array} -z \right]. \quad (19)$$

Where  $H_{p,q}^{m,n}[\cdot]$  denotes the Fox H-function.

Euler Beta transform,[8],

$$B[f(z):a,b] = \int_0^1 z^{a-1} (1-z)^{b-1} f(z) dz. \quad (20)$$

Laplace transform, ([9],equation3.1.1),

$$L[f(z):s] = \int_0^\infty e^{-sz} f(z) dz. \quad (21)$$

Mellin transform, ([9],equation 4.1.1),

$$M[f(z):s] = \int_0^\infty z^{s-1} f(z) dz = f^*(s), \text{Re}(s) > 0, \quad (22)$$

then,

$$f(z) = M^{-1}[f^*(s):x] = \int f^*(s) x^{-s} ds. \quad (23)$$

## 2 Main Results

1. Mellin-Barnes integral representation of p-k Mittag-Leffler function,  ${}_pE_{k,\alpha,\beta}^{\gamma,q}(z)$ .

**Theorem 2.1** Let  $k, p \in R^+ - \{0\}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, \text{and } q \in (0,1) \cup N$ , then the function  ${}_pE_{k,\alpha,\beta}^{\gamma,q}(z)$  is represented by the Mellin-Barnes integral as,

$${}_pE_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{kp^{-\frac{\beta}{k}}}{2\pi i \Gamma(\frac{\gamma}{k})} \int_L \frac{\Gamma(s)\Gamma((\frac{\gamma}{k})-qs)}{\Gamma((\frac{\beta}{k})-(\frac{\alpha s}{k}))} (-zp^{q-\frac{\alpha}{k}})^{-s} ds. \quad (24)$$

Where  $|arg z| < \pi$ ; the contour integration beginning at  $-i\infty$  and ending at  $+i\infty$ , and indented to separate the poles of the integrand as

$$s = -n \text{ for every } n \in N_0 \text{ (to the left) from those at } s = \frac{k}{q} + n \text{ for every } n \in N_0 \text{ (to the right).}$$

**Proof** Consider the integral on right side of equation(2.1) and use the theorem of calculus of residues,

$$A \equiv \frac{kp^{-\frac{\beta}{k}}}{2\pi i \Gamma(\frac{\gamma}{k})} \int_L \frac{\Gamma(s)\Gamma((\frac{\gamma}{k})-qs)}{\Gamma((\frac{\beta}{k})-(\frac{\alpha s}{k}))} (-zp^{q-\frac{\alpha}{k}})^{-s} ds$$

$= 2\pi i$  [sum of the residues at the poles  $s = 0, -1, -2, \dots$ ]

$$A \equiv \frac{kp^{-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} Re \left[ s \left( \frac{\Gamma(s)\Gamma((\frac{\gamma}{k})-qs)}{\Gamma((\frac{\beta}{k})-(\frac{\alpha s}{k}))} \right) (-zp^{q-\frac{\alpha}{k}})^{-s} \right]$$

$$A \equiv \frac{kp^{\frac{-\beta}{k}}}{\Gamma(\frac{\gamma}{k})} \sum_{n=0}^{\infty} \lim_{s \rightarrow -n} \frac{\pi(s+n)}{\sin \pi s} \frac{\Gamma((\frac{\gamma}{k})-qs)}{\Gamma(1-s)\Gamma((\frac{\beta}{k})-(\frac{\alpha s}{k}))} [(-zp^{\frac{q-\alpha}{k}})^{-s}]$$

$$kp^{\frac{-\beta}{k}} \sum_{n=0}^{\infty} \frac{\left(\frac{\gamma}{k}\right)_{qn}}{\Gamma((\frac{\beta}{k})+(\frac{\alpha n}{k}))} \frac{(zp^{\frac{q-\alpha}{k}})^n}{n!}$$

using equations (1.14) and (1.17), we have,

$$A \equiv {}_p E_{k,\alpha,\beta}^{\gamma,q}(z)$$

Hence.

#### Particular cases:

**Lemma 2.1.1** Let  $k, p \in R^+ - \{0\}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, and q \in (0,1) \cup N$ , and put  $p = k$  in equation(2.1), we have,

$${}_k E_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{k^{1-\frac{\beta}{k}}}{2\pi i \Gamma(\frac{\gamma}{k})} \int_L \frac{\Gamma(s)\Gamma((\frac{\gamma}{k})-qs)}{\Gamma((\frac{\beta}{k})-(\frac{\alpha s}{k}))} (-zk^{\frac{q-\alpha}{k}})^{-s} ds. \quad (25)$$

Where  $|arg z| < \pi$ ; the contour integration beginning at  $-i\infty$  and ending at  $+i\infty$ , and indented to separate the poles of the integrand as

$$s = -n \text{ for every } n \in N_0 \text{ (to the left) from those at } s = \frac{\frac{\gamma}{k} + n}{q} \text{ for every } n \in N_0 \text{ (to the right).}$$

Note: Result given by equation(2.2) is known result given by ([7], equation(25)).

**Lemma 2.1.2** Let  $k, p \in R^+ - \{0\}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, and q \in (0,1) \cup N$ , and put  $p = k = 1$  in equation(2.1), we have,

$${}_1 E_{1,\alpha,\beta}^{\gamma,q}(z) = \frac{1}{2\pi i \Gamma(\gamma)} \int_L \frac{\Gamma(s)\Gamma(\gamma-qs)}{\Gamma(\beta-\alpha s)} (-z)^{-s} ds. \quad (26)$$

Where  $|arg z| < \pi$ ; the contour integration beginning at  $-i\infty$  and ending at  $+i\infty$ , and indented to separate the poles of the integrand as

$$s = -n \text{ for every } n \in N_0 \text{ (to the left) from those at } s = \frac{\gamma + n}{q} \text{ for every } n \in N_0 \text{ (to the right).}$$

Note: Result given by equation(2.3) is known result given by ([1,7], equation(4.1.1) and (26)).

**Lemma 2.1.3** Let  $k, p \in R^+ - \{0\}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, and q \in (0,1) \cup N$ , and put  $p = k$  and  $q = 1$  in equation(2.1), we have,

$${}_k E_{k,\alpha,\beta}^{\gamma,1}(z) = \frac{k^{1-\frac{\beta}{k}}}{2\pi i \Gamma(\frac{\gamma}{k})} \int_L \frac{\Gamma(s)\Gamma((\frac{\gamma}{k})-s)}{\Gamma((\frac{\beta}{k})-(\frac{\alpha s}{k}))} (-zk^{\frac{1-\alpha}{k}})^{-s} ds. \quad (27)$$

Where  $|arg z| < \pi$ ; the contour integration beginning at  $-i\infty$  and ending at  $+i\infty$ , and indented to separate the poles of the integrand as

$$s = -n \text{ for every } n \in N_0 \text{ (to the left) from those at } s = \frac{\gamma}{k} + n \text{ for every } n \in N_0 \text{ (to the right).}$$

Note: Result given by equation(2.4) is known result given by ([7], equation(27)).

#### Relationship with Fox H-function

Let  $k, p \in R^+ - \{0\}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, and q \in (0,1) \cup N$ , then the function  ${}_p E_{k,\alpha,\beta}^{\gamma,q}(z)$  From equation(2.1), we have,

$$\begin{aligned}
 {}_p E_{k,\alpha,\beta}^{\gamma,q}(z) &= \frac{k p^{-\frac{\beta}{k}}}{2\pi i \Gamma(\frac{\gamma}{k})} \int_L \frac{\Gamma(s)\Gamma((\frac{\gamma}{k})-qs)}{\Gamma((\frac{\beta}{k})-\frac{\alpha s}{k})} (-zp^{q-\frac{\alpha}{k}})^{-s} ds, \\
 {}_p E_{k,\alpha,\beta}^{\gamma,q}(z) &= \frac{k p^{-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} H_{1,2}^{1,1} \left[ -zp^{q-\frac{\alpha}{k}} \mid \begin{array}{l} (\frac{\gamma}{k}, q); \\ (0,1), (1-\frac{\beta}{k}, \frac{\alpha}{k}); \end{array} \right].
 \end{aligned} \tag{28}$$

### Relationship with Wright hypergeometric function

Let  $k, p \in R^+ - \{0\}; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, and q \in (0,1) \cup N$ . The equation (1.6) can be written as (using equation(1.18)),

$${}_p E_{k,\alpha,\beta}^{\gamma,q}(z) = \frac{k p^{-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} \psi_1 \left[ \begin{array}{l} (\frac{\gamma}{k}, q); \\ (\frac{\beta}{k}, \frac{\alpha}{k}); \end{array} zp^{q-\frac{\alpha}{k}} \right]. \tag{29}$$

### 3 Integral Transform of ${}_p E_{k,\alpha,\beta}^{\gamma,q}(z)$ .

In this section we evaluate Euler Beta Transform, Laplace Transform and Mellin Transform of p-k Mittag-Leffler function,  ${}_p E_{k,\alpha,\beta}^{\gamma,q}(z)$ .

**Theorem 3.1** Let  $k, p \in R^+ - \{0\}; a, b, \sigma \in C; Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\sigma) > 0$  and  $q \in (0,1) \cup N$ , then Euler Beta Transform of p-k Mittag-Leffler function,  ${}_p E_{k,\alpha,\beta}^{\gamma,q}(z)$  is given by,

$$\int_0^1 z^{a-1} (1-z)^{b-1} {}_p E_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma) dz = \frac{k p^{-\frac{\beta}{k}} \Gamma(b)}{\Gamma(\frac{\gamma}{k})} {}_2\psi_2 \left[ \begin{array}{l} (\frac{\gamma}{k}, q), (a, \sigma); \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (a+b, \sigma); \end{array} xp^{q-\frac{\alpha}{k}} \right] \tag{30}$$

**Proof** Consider the left side integral and using equation (3.1), we have,

$$\begin{aligned}
 A &\equiv \int_0^1 z^{a-1} (1-z)^{b-1} {}_p E_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma) dz \\
 A &\equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k} x^n}{\Gamma_k(n\alpha + \beta)n!} \int_0^1 z^{\sigma n + a - 1} (1-z)^{b-1} dz,
 \end{aligned}$$

using equation(1.5), we have,

$$A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k} x^n}{\Gamma_k(n\alpha + \beta)n!} B(\sigma n + a, b)$$

using equation (1.15),(1.16) and (1.18), we have,

$$A \equiv \frac{kp^{-\frac{\beta}{k}}\Gamma(b)}{\Gamma(\frac{\gamma}{k})} {}_2\psi_2 \left[ \begin{matrix} (\frac{\gamma}{k}, q), (a, \sigma); \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (a+b, \sigma); \end{matrix} \middle| xp^{\frac{q-\alpha}{k}} \right]$$

Hence.

#### Particular cases:

**Lemma 3.1.1** Let  $k, p \in R^+ - \{0\}$ ;  $a, b, \sigma \in C$ ;  $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\sigma) > 0$  and  $q \in (0, 1) \cup N$ , and put  $p = k$ , in equation (3.1), we have Euler Beta Transform of p-k Mittag-Leffler function is,

$$\int_0^1 z^{a-1} (1-z)^{b-1} {}_k E_{k,\alpha,\beta}^{\gamma,q} (xz^\sigma) dz = \frac{k^{1-\frac{\beta}{k}}\Gamma(b)}{\Gamma(\frac{\gamma}{k})} {}_2\psi_2 \left[ \begin{matrix} (\frac{\gamma}{k}, q), (a, \sigma); \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (a+b, \sigma); \end{matrix} \middle| xk^{\frac{q-\alpha}{k}} \right] \quad (31)$$

Note: Equation (3.2) is known result given by ([7],equation (30)).

**Lemma 3.1.2** Let  $k, p \in R^+ - \{0\}$ ;  $a, b, \sigma \in C$ ;  $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\sigma) > 0$  and  $q \in (0, 1) \cup N$ , and put  $p = k = 1$ , in equation (3.1), we have Euler Beta Transform of p-k Mittag-Leffler function is,

$$\int_0^1 z^{a-1} (1-z)^{b-1} {}_1 E_{1,\alpha,\beta}^{\gamma,q} (xz^\sigma) dz = \frac{\Gamma(b)}{\Gamma(\gamma)} {}_2\psi_2 \left[ \begin{matrix} (\gamma, q), (a, \sigma); \\ (\beta, \alpha), (a+b, \sigma); \end{matrix} \middle| x \right] \quad (32)$$

Note: Equation (3.3) is known result given by ([7],equation (31)).

**Lemma 3.1.3** Let  $k, p \in R^+ - \{0\}$ ;  $a, b, \sigma \in C$ ;  $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\sigma) > 0$  and  $q \in (0, 1) \cup N$ , and put  $p = k$ , and  $q = 1$  in equation (3.1), we have Euler Beta Transform of p-k Mittag-Leffler function is,

$$\int_0^1 z^{a-1} (1-z)^{b-1} {}_k E_{k,\alpha,\beta}^{\gamma,1} (xz^\sigma) dz = \frac{k^{1-\frac{\beta}{k}}\Gamma(b)}{\Gamma(\frac{\gamma}{k})} {}_2\psi_2 \left[ \begin{matrix} (\frac{\gamma}{k}, 1), (a, \sigma); \\ (\frac{\beta}{k}, \frac{\alpha}{k}), (a+b, \sigma); \end{matrix} \middle| xk^{\frac{1-\alpha}{k}} \right] \quad (33)$$

Note: Equation (3.4) is known result given by ([7],equation (32)).

**Theorem 3.2** The Laplace transform of p-k Mittag-Leffler function,  ${}_p E_{k,\alpha,\beta}^{\gamma,q}(z)$ ,

$$\int_0^\infty z^{a-1} e^{-zs} {}_p E_{k,\alpha,\beta}^{\gamma,q} (xz^\sigma) dz = \frac{kp^{-\frac{\beta}{k}} s^{-a}}{\Gamma(\frac{\gamma}{k})} {}_2\psi_1 \left[ \begin{matrix} (\frac{\gamma}{k}, q), (a, \sigma); \\ (\frac{\beta}{k}, \frac{\alpha}{k}); \end{matrix} \middle| \frac{xp^{\frac{q-\alpha}{k}}}{s^\sigma} \right] \quad (34)$$

Where  $k, p \in R^+ - \{0\}$ ;  $a, \sigma \in C$ ;  $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\sigma) > 0$  and  $q \in (0, 1) \cup N$ , and  $|\frac{x}{s^\sigma}| < 1$ .

**Proof** Consider the right side integral and using equation(1.6), we have

$$A \equiv \int_0^\infty z^{a-1} e^{-zs} {}_p E_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma) dz$$

$$A \equiv \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p \Gamma_k(n\alpha + \beta)} \frac{x^n}{n!} \int_0^\infty z^{n\sigma+a-1} e^{-zs} dz$$

using definition of gamma function, we have

$$A \equiv s^{-a} \sum_{n=0}^{\infty} \frac{{}_p(\gamma)_{nq,k}}{{}_p \Gamma_k(n\alpha + \beta) n!} \Gamma(\sigma n + a) \left( \frac{x}{s^\sigma} \right)^n$$

using equation (1.15),(1.17), and (1.18), we have

$$A \equiv \frac{kp^{\frac{\beta}{k}} s^{-a}}{\Gamma(\frac{\gamma}{k})} {}_2\psi_1 \left[ \begin{matrix} (\frac{\gamma}{k}, q), (a, \sigma); \\ (\frac{\beta}{k}, \frac{\alpha}{k}); \end{matrix} \middle| \frac{xp^{\frac{q-\alpha}{k}}}{s^\sigma} \right]$$

Hence.

#### Particular cases:

**Lemma 3.2.1** Let  $k, p \in R^+ - \{0\}$ ;  $a, \sigma \in C$ ;  $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\sigma) > 0$  and  $q \in (0,1) \cup N$ , and put  $p = k$ , in equation (3.5), we have Laplace Transform of p-k Mittag-Leffler function is,

$$\int_0^\infty z^{a-1} e^{-zs} {}_k E_{k,\alpha,\beta}^{\gamma,q}(xz^\sigma) dz = \frac{s^{-a} k^{1-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} {}_2\psi_1 \left[ \begin{matrix} (\frac{\gamma}{k}, q), (a, \sigma); \\ (\frac{\beta}{k}, \frac{\alpha}{k}); \end{matrix} \middle| \frac{x}{s^\sigma} k^{\frac{q-\alpha}{k}} \right] \quad (35)$$

Note: Equation (3.6) is known result given by ([7],equation (34)).

**Lemma 3.2.2** Let  $k, p \in R^+ - \{0\}$ ;  $a, b, \sigma \in C$ ;  $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\sigma) > 0$  and  $q \in (0,1) \cup N$ , and put  $p = k = 1$ , in equation (3.5), we have Laplace Transform of p-k Mittag-Leffler function is,

$$\int_0^\infty z^{a-1} e^{-zs} {}_1 E_{1,\alpha,\beta}^{\gamma,q}(xz^\sigma) dz = \frac{s^{-a}}{\Gamma(\gamma)} {}_2\psi_1 \left[ \begin{matrix} (\gamma, q), (a, \sigma); \\ (\beta, \alpha); \end{matrix} \middle| \frac{x}{s^\sigma} \right] \quad (36)$$

Note: Equation (3.7) is known result given by ([7],equation (34)).

**Lemma 3.2.3** Let  $k, p \in R^+ - \{0\}$ ;  $a, b, \sigma \in C$ ;  $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(\sigma) > 0$  and  $q \in (0,1) \cup N$ , and put  $p = k$ , and  $q = 1$  in equation (3.5), we have Laplace Transform of p-k Mittag-Leffler function is,

$$\int_0^\infty z^{a-1} e^{-zs} {}_k E_{k,\alpha,\beta}^{\gamma,1}(xz^\sigma) dz = \frac{s^{-a} k^{1-\frac{\beta}{k}}}{\Gamma(\frac{\gamma}{k})} {}_2\psi_1 \left[ \begin{matrix} (\frac{\gamma}{k}, 1), (a, \sigma); \\ (\frac{\beta}{k}, \frac{\alpha}{k}); \end{matrix} \middle| \frac{x}{s^\sigma} k^{\frac{1-\alpha}{k}} \right] \quad (37)$$

Note: Equation (3.8) is known result given by ([7],equation (35)).

**Theorem 3.3** The Mellin transform of p-k Mittag-Leffler function,  ${}_p E_{k,\alpha,\beta}^{\gamma,q}(z)$ ,

$$\int_0^\infty t^{s-1} {}_p E_{k,\alpha,\beta}^{\gamma,q}(-wt) dz = \frac{\Gamma(s)\Gamma(\frac{\gamma}{k}-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(\frac{\gamma}{k})} \left(\frac{p^{\frac{\alpha}{k}-q}}{w}\right)^s \quad (38)$$

Where  $k, p \in R^+ - \{0\}$ ;  $\alpha, \beta, \gamma, s \in C$ ;  $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(s) > 0$  and  $q \in (0,1) \cup N$ .

**Proof** Put  $z = -wt$  in equation (2.1), we have,

$$\begin{aligned} {}_p E_{k,\alpha,\beta}^{\gamma,q}(-wt) &= \frac{kp^{-\frac{\beta}{k}}}{2\pi i \Gamma(\frac{\gamma}{k})} \int_L \frac{\Gamma(s)\Gamma(\frac{\gamma}{k}-qs)}{\Gamma(\frac{\gamma}{k})\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})} (-wtp^{q-\frac{\alpha}{k}})^{-s} ds \\ {}_p E_{k,\alpha,\beta}^{\gamma,q}(-wt) &= \frac{kp^{-\frac{\beta}{k}}}{2\pi i \Gamma(\frac{\gamma}{k})} \int_L f^*(s)(t)^{-s} ds. \end{aligned} \quad (39)$$

where,

$$f^*(s) = \frac{\Gamma(s)\Gamma(\frac{\gamma}{k}-qs)}{\Gamma(\frac{\gamma}{k})\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})} (-wtp^{q-\frac{\alpha}{k}})^{-s}$$

using equation (1.22),(1.23) and (3.10), which immediately leads to (3.9).

#### Particular cases:

**Lemma 3.3.1** Let  $k, p \in R^+ - \{0\}$ ;  $\alpha, \beta, \gamma, s \in C$ ;  $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(s) > 0$  and  $q \in (0,1) \cup N$ . put  $p = k$ ,

Then Millin transform of generalized Mittag-Leffler function,  ${}_p E_{k,\alpha,\beta}^{\gamma,q}(z)$  is,

$$\int_0^\infty t^{s-1} {}_k E_{k,\alpha,\beta}^{\gamma,q}(-wt) dz = \frac{\Gamma(s)\Gamma(\frac{\gamma}{k}-qs)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})} \left(\frac{k^{\frac{\alpha}{k}-q}}{w}\right)^s. \quad (40)$$

Note: equation (3.11) is known result given by ([7],equation(36)).

**Lemma 3.3.2** Let  $k, p \in R^+ - \{0\}$ ;  $\alpha, \beta, \gamma, s \in C$ ;  $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(s) > 0$  and  $q \in (0,1) \cup N$ . put  $p = k = 1$ , Then Millin transform of generalized Mittag-Leffler function  ${}_p E_{k,\alpha,\beta}^{\gamma,q}(z)$  is,

$$\int_0^\infty t^{s-1} {}_1 E_{1,\alpha,\beta}^{\gamma,q}(-wt) dz = \frac{\Gamma(s)\Gamma(\gamma-qs)}{\Gamma(\beta-\alpha s)\Gamma(\gamma)} \left(\frac{1}{w}\right)^s. \quad (41)$$

Note: equation (3.12) is known result given by ([7],equation(38)).

**Lemma 3.3.3** Let  $k, p \in R^+ - \{0\}$ ;  $\alpha, \beta, \gamma, s \in C$ ;  $Re(\alpha) > 0, Re(\beta) > 0, Re(\gamma) > 0, Re(s) > 0$  and  $q \in (0,1) \cup N$ . put  $q = 1$ ,

Then Millin transform of generalized Mittag-Leffler function  ${}_p E_{k,\alpha,\beta}^{\gamma,q}(z)$  is,

$$\int_0^\infty t^{s-1} {}_k E_{k,\alpha,\beta}^{\gamma,1}(-wt) dz = \frac{\Gamma(s)\Gamma(\frac{\gamma}{k}-s)}{\Gamma(\frac{\beta}{k}-\frac{\alpha s}{k})\Gamma(\frac{\gamma}{k})} \left(\frac{k^{\frac{\alpha}{k}-1}}{w}\right)^s \quad (42)$$

Note: equation (3.13) is known result given by ([7],equation(39)).

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