Semi-open (S-open) and semi-closed (S-closed) mapping on a TAL

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Abstract: To introduce characterisations and properties of S-continuous, S-open and S-closed mappings on to TAL. These concepts and relevant theory are obtained under the case of no order-reversing involution. They may be useful tool for establishing the theory of semi- topological atomistic lattices.

Keewords: Remote neighbourhood (RN), Topological Automistic Lattice (TAL), Semi interior, S-separations, Maximal Point.

Introduction

Cameron and Woods (1987) introduced the Concepts of S-Continuous mappings and S-open mappings. They investigated the properties of these mappings and their relationships to the properties of semi-open sets. Khan and Ahmad (2001) further worked on the characterisations and properties of S-continuous, S-open and S-closed mappings.

In genral topology or fuzzy topology we know that the concepts of semi-open and semi-closed sets depend on the notion of open sets or fuzzy open sets. It is known to us traditional neighbourhood (nhd) method is no longer effective in fuzzy topology. To overcome this difficulty, Pu and Liu (1980) introduced Q-neighbourhood (Q-nhd) and found it to be effective. It should be noticed that in lattice theory there may be no order- reversing involution and hence one can't define a Q-nhd. For this reason Wang Guojan [89] introduced the concept of remote nhd (RN) in Lattice theory, as a generalisation of Q-nhd. Therefore, in lattice theory the remote nhd method and Cotopology take main role Guojan (1989). Therefore how to generalise these concepts of semi-open, semi-closed set on a topological atomistic lattice (TAL) is a fairly significant work.

Definition 1.2: [Chang 1968] A fuzzy topology on a non- empty set X is a family T of fuzzy subsets of X which satisfies the following three conditions

I. $\phi, x \in$ T

II. A, B $\in$ T $\Rightarrow$ A $\wedge$ B $\in$ T

III. $A_i \in$ T for each i $\in$ I $\Rightarrow \vee A_i \in$ T

The pair (X, T) is called a fuzzy topological space or $f + S$. Every member of T is called T-Open fuzzy set (or simply open fuzzy set). A fuzzy set is T-closed iff its complement A^C (or 1- A) is T-open.

As in general topology, the indiscrete fuzzy contains ϕ and X, while the discrete fuzzy topology all fuzzy sets.

Definition 1.3: [Pu and Lie, 1980]. The fuzzy subset x_a of a non- empty set X with $x \in X$ and $0 < a \leq 1$ defined by

$$x_a(p) = \begin{cases} a \text{ if } p = x \\ 0, \text{ otherwise} \end{cases}$$

is called a fuzzy point in X with support x and value a.

In this chapter we shall always consider L, a complete distributive atomistic lattice [Maeda & Maeda, 1970] unless or otherwise stated and L_o, the subset of the set of all atoms of L so that every element in L can be represented as a join of elements in L_o

Definition 1.4: [Jha and Parhi, 2003] Let T $\subseteq L$, 0, 1$\in$ T is closed under the operation of finite join and arbitrary meet. Then we call T, a Co- topology on L and elements of T will be called closed elements. If $a \in L$ and $n \in$ T and $a < n$, then n is called remote nhd (RN) of a. All remote nhds of a will be denoted by n (a) and n (a) is a base on L. The pair (L, T) is called a topological atomistic lattice (TAL). Wang Guojun) [1989] introduced closure operator in the lattice theory in the light of the definition of that of J. C. C. Meckinscy and A. Tarki [1947], Lal and Jha [1975, 1983], Jhs and Singh [2008],

Definition 1.5: [Jha & Parhi 2003]. Let (L, T) be a TAL and $x \in L$. Then the meet of all closed elements which contain x is called the closure of x and will be denoted by $\bar{x}$. An element $c \in L$ is called component of $a \in L$ if c immediately preceeds a, i.e., for every $b \in L$, $c \leq b \leq a \Rightarrow b = c$. The set of all component of $a \in L$ is denoted by π_a. The component of 1 is called the maximal point of L and the set of all maximal points of L is denoted by π_L. Let L be a complete lattice and $a \in L$, B $\subseteq L$, B is called minimal family a if

(1) Sup B = a and

(2) A $\subseteq L$ and $a \leq$ Sup A $\Rightarrow$ for each $x \in$ B there exists a z $\in$ A such that $x \leq$ z. For each $a \in L$, the greatest Minimal family B ($\subseteq L$) will be called the standard minimal family a and will be denoted by $B^*(a)$

Definition 1.6: [Jha and Parhi, 2003] Let L_1 and L_2 be two complete atomistic distributive lattices. Let $f: L_1 \rightarrow L_2$ be a mapping. Then f is called from L_1 to L_2 if

(1) $f(a) = 0$ iff $a = 0$

(2) f is join preserving

(3) f^{-1} is join preserving, where $f^{-1}: L_2 \rightarrow L_1$ is defined by

$$f^{-1}(y) = \vee \{x \in L_1: f(x) \leq y\} \text{ for each } y \in L_2,$$

The following are the results proved by Jhs and Parhi (2003) in a TAL (L, T)

(1) Let $a \in L_1$, $a \neq$ o. Then for each $b \in L_1$, $b \leq a$, there exists a component c of a with $b \leq a$. Further more, there exists a unique component c of a with $b \leq c$ iff the different components of a are disjoint.

(2) For each $a \in L$, the different components of a are disjoint iff L is up-total order, i.e., for each $b \in L$

$$\uparrow b = \{x \in L: b \leq x\}$$

is a total order set about $\leq$

(3) For each $x \in L$, $a \in L$ is called an interior point of x iff there exists $x \in n(a)$ such that $n \vee x = 1$. Define $x^0 = \vee \{a \in L : a$ is an interior points of $x\}$ and we call x^0 the interior of x. Obviously $a^0 \leq a$.

(4) If a is an interior of x with $a < b$ $(a, b \in L)$. Then b is called an interior point of x.

(5) For each x, y $\in L$,
(I) $1^0 = 1$, $0^0 = 0$
(II) $x \leq y \Rightarrow x^0 \leq y^0$
(III) $(x^0)^0 = x^0$
(IV) If L is up–directed, then $(x \wedge y)^0 = x^0 \wedge y^0$

(6) An element $x \in L$ is called a semi-closed element if for each t $\in$ T with t $\vee x \neq 1$, there is an s $\in$T- {1} such that $x \vee$ t $\leq$ s. The set of all semi-closed elements is denoted by SC (T). Let $a \in L$, t $\in$ SC (T) and a $\leq$ t. Then t is called a semi-remote nhd (SRN) of a. All the semi-nhds of a will be denoted by $n^*(a)$
In a TAL (L,T):
I. A closed elements is semi-closed.
II. The meet of an arbitrary family of semi-closed elements is a semi-closed element.
III. The join of two semi-closed elements is semi-closed. Further the join of a semi-closed and a closed element is semi-closed and also the meet of them is a semi-closed element.

(7) Putting F (T) = $\{x \in$ SC (T): for each s $\in$ SC (T) and $x \vee$ s $\in$ SC (T)$\}$ we observe that F (T) is cotopology on L and T$\subseteq$ F (T).

(8) Let (L, T) be a TAL. If T_1 is an other topology L and SC(T_1) = SC(T). Then $T_1 \subseteq$ F (T). For each t $\in$ SC (T)-{1}, there exists an s $\in$ T-{1} such that t $\leq$ s. Moreover SC (F (T)) = SC (T). F (T) is the finest Cotopology among those Cotopologies on L which generates the same SC (T). The semiclosure of an element x in a topological atomistic lattice (L, T) is defined as

$$\mathrm{Scl}x = \wedge \{y \in \mathrm{SC}(T): x \leq y\}$$

The following properties hold:

I The Scl$x$ is a semi-closed element for each $x \in L$
II An element x is semi-closed iff Scl$x = x$
III $x \leq$ Scl$x$ for each $x \in L$
IV Scl = 1, Scl = 0
V Scl (Scl x) = Scl x
VI Scl $(x \vee y) \geq$ Scl $(x) \vee$ Scl (y)
VII $\overline{Sclx} = S\overline{clx} = \overline{x}$
VIII When one of x and y is closed,

$$\mathrm{Scl}(x \vee y) = \mathrm{Scl}\, x \vee \mathrm{Scl}\, y.$$

(9) Let (L, T) be a TAL and $a \in L$, then a is called an s-adherence point of x (or semi-adherence point of x) if for each $t \in n^*(a)$, $x \nleq$ t. If a is an s-adherence point of x and a $\nleq x$ or $a \leq x$ but for $b \in L$ such that $a \leq b \leq x$. We have $x \nleq b \vee$ t, then a is called an s-accumulation point of x. The join of all s-accumulation of x is called the semi derived element of x and will be denoted by ds$x$. We have
I. $a \leq$ Scl x iff a is an s-adherence of x
II. Scl $x = \vee\{a$: a is an s-adherence of $x\}$
III. Scl $x = x \vee$ ds$x$.
IV. For a semi-closed element y $\in L$, if $y^0 \leq x \leq$ y then x is a semi-closed element.
V. Let (L, T) be a TAL with up-total order and $x \in L$, be a semi-closed element. Then $(\overline{x})^0 \leq \overline{x} \leq x$
VI. Let (L, T) be up-total order and $x \in L$, x is semi-closed element iff there exists a closed element y such that

$$y^0 \leq x \leq y$$

10. x is semi-closed iff $(\overline{x})^0 \leq x$ Also x is semi-closed iff $(x)^0 \leq x$

Definition 3.1: Let (L_1, T_1) and (L_2, T_2) be two TALs. Then a mapping f: $L_1 \rightarrow L_2$ is said be S-open (res. S-closed) if the image of every semi-open (res. semi-closed) element in L_1 is semi-open (res. semi-closed) in L_2.

Obviously a semi-open (S-open) function is open.

Next, we define

Definition 3.2: An element $\flat \in L$ is called a boundary element of an element $a \in L$ if and only $\flat \in \overline{a} \wedge \overline{a}^c$. The union of all the boundary elements of a is called a boundary of a and is denoted by Bda. It is clear that

$$Bda = \overline{a} \wedge \overline{a^c}$$

Where (L, T) be a TAL.

We now introduce the notion semi boundary of an element a in a TAL

Definition 3.3: *Semi boundary (briefly s Bd) of an element a in a TAL (L,T) is defined as*

$$\mathrm{S\,Bda} = \mathrm{Scl}\, a \wedge \mathrm{Scl}\, a^c$$

In the following, we characterise S-open mappings in terms of Sint, Scl, SBd

Theorem 3.4: Let (L_1, T_1) and (L_2, T_2) be two TALs For a function f: $L_1 \rightarrow L_2$ the following statements are equivalent for every $a \in L_1$ and $b \in L_2$:

(1) f is S-open
(2) f (Sint a) $\leq (f(a))^0$
(3) Sint$f^{-1}(b) \leq f^{-1}(b^0)$
(4) $f^{-1}(\overline{b}) \leq \mathrm{Scl}\, f^{-1}(b)$

(5) $f^{-1}(\text{Bdb}) \leq \text{SBd}\,(f^{-1}(b))$

Proof : (1) $\Rightarrow$ (2) Obviously f (Sint a) $\leq f$ (a). f is S-open gives f (Sint a) that is open in L_2. But $(f\,(a))^0$ is the largest open element such that $(f\,(a))^0 \leq f\,(a)$. Therefore f (Sint a) $\leq (f\,(a))^0$ for any $a \in L_1$. This gives (2)

(2) $\Rightarrow$ (3) For any b $\in L_2$, $f^{-1}(\text{b}) \in L_1$. Then by (2)

$$f\,(\text{Sint}\, f^{-1}(\text{b})) \leq (f(f^{-1}(\text{b}))^0 \leq \text{b}^0$$

or f (Sint f^{-1} (b)) $\leq \text{b}^0$ or Sint f^{-1} (b)

$$\leq f^{-1} f\,(\text{Sint}\, f^{-1}(\text{b})) \leq f^{-1}(\text{b}^0)$$

or Sint f^{-1} (b) $\leq f^{-1}(\text{b}^0)$ This gives (3)

(3) $\Rightarrow$ (4). By (3), we have

$$\text{Sint}\, f^{-1}(\text{b}) \leq f^{-1}(\text{b}^0)$$

$$(f^{-1}(\text{b}^0))^C \leq (\text{Sint}\, f^{-1}(\text{b}))^C$$

$$\leq \text{Scl}\,(f^{-1}(\text{b}))^0$$

Or $f^{-1}(b^0)^c \leq \text{Scl}\, f^{-1}(b)^c$

Or $f^{-1}(\overline{b^c}) \leq \text{Scl}\, f^{-1}(b^c)$

Or $f^{-1}(\overline{x}) \leq \text{Scl}\, f^{-1}(x)$, where $x = \text{b}^c$

which is an element of L_2. This gives (4)

(4) $\Rightarrow$ (5) For b $\in L_2$

$Bdb = \overline{b} \wedge \overline{b^c}$ Bdb = b $\wedge$ b^c is closed in L_2

Now $f^{-1}(Bdb) = f^{-1}(\overline{b}) \wedge f^{-1}(\overline{b^c})$,

Using (4), we have

$$f^{-1}(\text{Bdb}) \leq \text{Scl}\,(f^{-1}(\text{b})) \wedge \text{Scl}\,(f^{-1}(\text{b}^c))$$

or $f^{-1}(\text{Bdb}) \leq \text{Scl}\, f^{-1}(\text{b}) \wedge \text{Scl}\,(f^{-1}(\text{b}))^c = \text{SBd}\, f^{-1}$ (b)

This gives (5)

In following, we give characterisation of S-closed mapping as follows:

Theorem 3.5: Let (L_1, T_1) and (L_2, T_2) be two TALs. A function f: $L_1 \rightarrow L_2$ in S-closed if and only if $\overline{f(x)} \leq f(\text{Scl}\, x)$, for each $x \in L_1$

Proof: Obviously f (x) $\leq f$ (Scl x), f (Scl x) is closed, since f is semi-closed. But $\overline{f(x)}$ is the smallest set with $f(\overline{x}) \leq f(x)$. Therefore, $f(x) \leq f(\overline{\text{Scl}x})$

Conversely, $x \in L_1$ is a semi-closed element

$\Rightarrow f$ (x) is closed

By hypothesis $f(x) \leq f(\overline{\text{Scl}x}) = f(x)$ or $f(\overline{x}) \leq f(x)$.

This proves that f (x) is closed.

Theorem 3.6: If a function f: $L_1 \rightarrow L_2$ is S-closed, then for each b in a TAL (L_2, T_2) and semi-open element u in a TAL (L_1, T_1) with $u \geq f^{-1}$ (b), there exists an open element v in L_2 with $v \geq$ b and f^{-1} (v) $\leq u$.

Proof: Let u be an arbitrary semi-open element in L_1 with u $\geq f^{-1}$ (b), where b $\in L_2$. clearly $(f\,(\text{b}^c))^c = v$ (say) is open in L_2. Since f^{-1} (b) $\leq$ u, then straight forward calculation give that b $\leq v$. Moreover, we have

$$f^{-1}(v) = f^{-1}(f\,(u^c))^c = (f^{-1} f\,(u^c))^c \leq u$$

or $f^{-1}(v) \leq u$.

Theorem 3.7: Let f: $L_1 \rightarrow L_2$ be a surjective function from *a* TAL *(L_1, T_1)* to a TAL *(L_2, T_2)*. If for each $b \in L_2$, and each semi – open element u with $u \geq f^{-1}(b)$, there exists an open element $v \in L_2$ with $v \geq b$ such that $f^{-1}(v) \leq u$, then f is S-closed.

Proof: Let ω be an arbitrary semi-closed element in L_1 and $y \in (f(\omega))^c$. Then

$$f^{-1}(y) \leq f^{-1}(f(\omega))^c = (f^{-1}f(\omega))^c \leq \omega^c$$

or $f^{-1}(y) \leq \omega^c$.

Since ω^c is semi-open, therefore there exists an open element v_y with $y \in v_y$ such that $f^{-1}(v_y) \leq \omega^c$. Since f is subjective, we have

$$y \in v_y \leq (f\,(\omega))^c.$$

Thus $(f\,(\omega)^c) = V\{v_y \mid y \in (f\,(\omega)^c)\}$ is open in L_2 or (f (ω) is closed in L_2. Taking $v_y = v$, it is proved that f is S-closed.

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REFERENCES

[1] Abbas, S.E: International J. of Contemporary Mathematical Sciences, Vol. 3, No. 33-36, (2003), 2689-2700.
[2] Ahmad, B. and Kharal, M.A. : J. of Fuzzy Math. Vol. 16, No. 2, (2008), 341-349.
[3] Zadeh L.A : Fuzzy Sets, Information and Control, Vol. 8 (1965), 338-353.
[4] Athar, M. and Ahmad, B. : Advance in Fuzzy System, Vol. (2008), Article ID 586893, p. 9.
[5] Chavg, C.L. : J. Math. Anal. Appl. 24 (1968), 182-190
[6] Cameron, D. E. and Woods, G. : S-Continuous and S-open mappings, 1987.
[7] Cao, J.; Ganster, M. and Reilly, I.: Housten J. of Math. Vol. 24, No 4 (1998) 681-688
[8] Cueva, M.C. and Saraf, R.K. : Proyecciones, Vol. 20, No2 (2001), 127-138
[9] Du, S. Qin, Q.; Wang, Q; and Li, B. : In Proceedings of the 1stInternational Conferense on Advances in Natural Computation (ICNC' 05) Vol. 3612 of Lecture Notes in Computer Science, 1261-1273 (2005).
[10] Egenhofer, M.J. and Fronzosa, R.D. : International J. of Geographical Information Systems Vol 5, No. 2 (1991), 161-174.
[11] Jha, J.S. and Lal, R.N. : J. of Natural Sciences, Vol. 8, No. 1 (1975), 75-77.

[12] Jha, J.S. and Singh, M. K: Anusandhan, vol 9, No 16 (2008), 27-31
[13] Jha, J.S. and Parhi, K. : Aryabhat Research J. of Phipical Sciences, Vol. 6, No 1-2 (2003), 39-41
[14] Kham M. and Ahmad, B. : Punjab Unit: Journal, Vol 34, No. 1 (2001) 107-114.
[15] Lee, S.J. and Lee, E.P. : International J. of Mathematics and Mathematical Sciences Vol. 27, No. 1 (2001), 53-63.
[16] Maeda, F. and Maeda, S: The theory of Symmetric Lattices, Springer-Verlag Publication (1970).
[17] Naschie El, M.S. : Chaos, Soliton & Fractals, Vol. 9, No 3 (1998) 517-529
[18] Tang X: ITC Dissertation no. 108, University of Twente, The Netherlands (2004).
[19] Zahran, A.M. : Fuzzy sets and systems Vol. 116, No.3 (2000) 353-359.
[20] Jha, J.S. and Wats : Anusandhan, Vol. XII, No 22 (June 2010), 111-116.