Estimation of Error in Runge-Kutta Fehlberg's Method

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Abstract: Five ordinary equations with boundary conditions have been solved with the help of Runge-Kutta Fehlberg's method and compared with the exact values. It has been observed that the Runge-Kutta Fehlberg's method is accurate up to 10 digits after the decimal point at some points. Average of maximum errors indicates that Runge-Kutta Fehlberg's method is at least accurate up to two digits after the decimal point. The average of maximum errors and maximum percentage errors has been found to be 0.0011957857183 and 0.0733067638714 respectively.

Key words: Runge-Kutta Fehlberg's method, error, average, ordinary differential equation, initial value problem.

Introduction

One way to guarantee accuracy in the solution of an initial value problem (I.V.P.) is to solve the problem twice using step sizes h and h/2 and compare answers at the mesh points corresponding to the larger step size. But this requires a significant amount of computation for the smaller step size and must be repeated if it is determined that the agreement is not good enough. The Runge-Kutta Fehlberg's method (denoted by RKF45) is one way to try to resolve this problem. It has a procedure to determine if the proper step size h is being used. At each step, two different approximations for the solution are made and compared. If the two answers are in close agreement, the approximation is accepted. If the two answers do not agree to a specified accuracy, the step size is reduced. If the answers agree to more significant digits than required, the step size is increased.[1-10]

Each Runge-Kutta-Fehlberg step requires the use of the following six values:

$$k_1 = h f(t_j, y_j)$$

$$k_2 = h f\left(t_j + \frac{1}{4}h,\ y_j + \frac{1}{4}k_1\right)$$

$$k_3 = h f\left(t_j + \frac{3}{8}h,\ y_j + \frac{3}{32}k_1 + \frac{9}{32}k_2\right)$$

$$k_4 = h f\left(t_j + \frac{12}{13}h,\ y_j + \frac{1932}{2197}k_1 - \frac{7200}{2197}k_2 + \frac{7296}{2197}k_3\right)$$

$$k_5 = h f\left(t_j + h,\ y_j + \frac{439}{216}k_1 - 8k_2 + \frac{3680}{513}k_3 - \frac{845}{4104}k_4\right)$$

$$k_6 = h f\left(t_j + \frac{1}{2}h,\ y_j - \frac{8}{27}k_1 + 2k_2 - \frac{3544}{2565}k_3 + \frac{1859}{4104}k_4 - \frac{11}{40}k_5\right)$$

Then an approximation to the solution of the I.V.P. is made using a Runge-Kutta method of order 4:

$$y_{j+1} = y_j + \frac{25}{216}k_1 + \frac{1408}{2565}k_3 + \frac{2197}{4104}k_4 - \frac{1}{5}k_5$$

And a better value for the solution is determined using a Runge-Kutta Fehlberg's method :[10-16]

$$z_{j+1} = y_j + \frac{16}{135}k_1 + \frac{6656}{12825}k_3 + \frac{28561}{56430}k_4 - \frac{9}{50}k_5 + \frac{2}{55}k_6$$

The optimal step size sh can be determined by multiplying the scalar s times the current step size h. The scalar s is

$$s = \left(\frac{\epsilon h}{2\,|z_{j+1} - y_{j+1}|}\right)^{1/4} = 0.840896\left(\frac{\epsilon h}{|z_{j+1} - y_{j+1}|}\right)^{1/4}$$

where ϵ is the specified error control tolerance.

Material and Method

We have solved the differential equations given in Table-1 at the points
$0=x_0, x_1, x_2, \ldots\ldots, x_n=1, n=100, x_i - x_{i-1} = 0.01, i=1, 2, \ldots, 100$
with the help of Runge-Kutta Fehlberg's method by using the following computer program written in C++ programming language.

With the help of computer programs, we have calculated the value of y, exact value of y, error in the calculated values of y and percentage error in the calculated value of y. We have calculated the values of y in the interval [0, 1] by assuming the step size of 0.01. Percentage error in the calculated value of y is defined as

$$PE_y = \frac{\text{Difference between calculated and exact values of y}}{\text{Exact value of y}} \times 100$$

Table-1: Differential equations with boundary conditions along with exact solution

S. No.	Differential Equation	Boundary Condition	Exact Solution
1	$dy/dx = -y$	y=1 when x=0	$y=e^{-x}$
2	$dy/dx = 3x^2 + y$	y=1 when x=0	$y=-3x^2-6x-6+2e^x$
3	$dy/dx = x$	y=0 when x=0	$y=x^2/2$
4	$dy/dx = x + xy$	y=1 when x=0	$y=2e^{x2/2} - 1$
5	$dy/dx = 1 + x^2$	y=0 when x=0	$y=x + x^3/3$

Result and Discussion

DE1: Error in the Solution of Differential equation dy/dx = -y, y=1 when x=0, as obtained by Runge-Kutta Fehlberg's method

Differential equation has been solved by Runge-Kutta Fehlberg's method with the help of computer program developed in C++ programming language at different values of x. Exact values of y at these points have also been calculated. Exact value of y, calculated value of y by Runge-Kutta Fehlberg's method and difference between them (called error in the value of y) at different points is included in Table-2. A close look of Table-2 indicates the following-

Maximum Error	=0.0007320047104
Minimum Error	=0.0000000000482
Maximum Percentage Error	=0.1989795102678
Minimum Percentage Error	=0.0000000052723

It is clear that the error may occur in the value of y calculated by Runge-Kutta Fehlberg's method after 3 places of decimal point. Graph between errors in the value of y calculated by Runge-Kutta Fehlberg's method at different points is shown in Graph-1. Graph between percentage errors in the value of y calculated by Runge-Kutta Fehlberg's method at different points is shown in Graph-2.

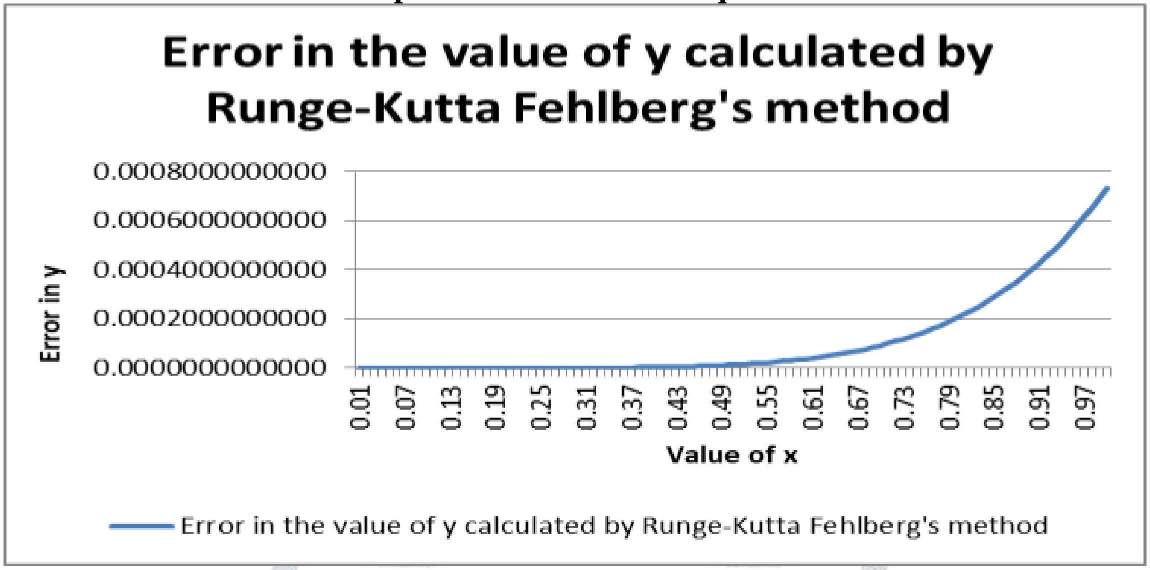
Table-2: Exact value of y, calculated value of y by Runge-Kutta Fehlberg's method , error in the value of y calculated by Runge-Kutta Fehlberg's method and percentage error in the value of y calculated by Runge-Kutta Fehlberg's method at different points of interval [0,1] for differential equation dy/dx = -y, y=1 when x=0

Value of x	Exact value of y for dy/dx = -x, y=1 when x=0	Value of y calculated by Runge-Kutta Fehlberg's method	Error in the value of y calculated by Runge-Kutta Fehlberg's method	Percentage error in the value of y calculated by Runge-Kutta Fehlberg's method
0.01	0.9900498339705	0.9900498341920	0.0000000002216	0.0000000223801
0.02	0.9801986737449	0.9801986735264	0.0000000002185	0.0000000222910
0.03	0.9704455341992	0.9704455346703	0.0000000004711	0.0000000485415
0.04	0.9607894400113	0.9607894402826	0.0000000002712	0.0000000282290
0.05	0.9512294273356	0.9512294264087	0.0000000009269	0.0000000974444
0.06	0.9417645348473	0.9417645338981	0.0000000009492	0.0000001007863
0.07	0.9323938196281	0.9323938180425	0.0000000015856	0.0000001700523
0.08	0.9231163480373	0.9231163505531	0.0000000025158	0.0000002725302
0.09	0.9139311888121	0.9139311887639	0.0000000000482	0.0000000052723
0.10	0.9048374234292	0.9048374236231	0.0000000001939	0.0000000214286
0.11	0.8958341358305	0.8958341372500	0.0000000014195	0.0000001584575
0.12	0.8869204390961	0.8869204358515	0.0000000032446	0.0000003658274
0.13	0.8780954351076	0.8780954317962	0.0000000033115	0.0000003771179
0.14	0.8693582348806	0.8693582280166	0.0000000068640	0.0000007895471
0.15	0.8607079841204	0.8607079771676	0.0000000069528	0.0000008077985
0.16	0.8521437920137	0.8521437826888	0.0000000093249	0.0000010942906
0.17	0.8436648150878	0.8436647912544	0.0000000238334	0.0000028249826
0.18	0.8352702178835	0.8352701824867	0.0000000353968	0.0000042377662
0.19	0.8269591359150	0.8269590952246	0.0000000406904	0.0000049204874
0.20	0.8187307628380	0.8187307099687	0.0000000528693	0.0000064574765
0.21	0.8105842512848	0.8105841804548	0.0000000708300	0.0000087381427
0.22	0.8025187989192	0.8025186990717	0.0000000998475	0.0000124417633
0.23	0.7945336110278	0.7945334751564	0.0000001358714	0.0000171007732
0.24	0.7866278652864	0.7866277044724	0.0000001608140	0.0000204434642
0.25	0.7788007830714	0.7788005721513	0.0000002109201	0.0000270826765
0.26	0.7710515931569	0.7710513224507	0.0000002707062	0.0000351086997
0.27	0.7633795088972	0.7633791834926	0.0000003254045	0.0000426268382
0.28	0.7557837405548	0.7557833278735	0.0000004126812	0.0000546030843
0.29	0.7482635738226	0.7482630749882	0.0000004988344	0.0000666656015
0.30	0.7408182339286	0.7408176244737	0.0000006094548	0.0000822677957
0.31	0.7334469544756	0.7334461998044	0.0000007546712	0.0001028937645
0.32	0.7261490422675	0.7261481355521	0.0000009067154	0.0001248662930

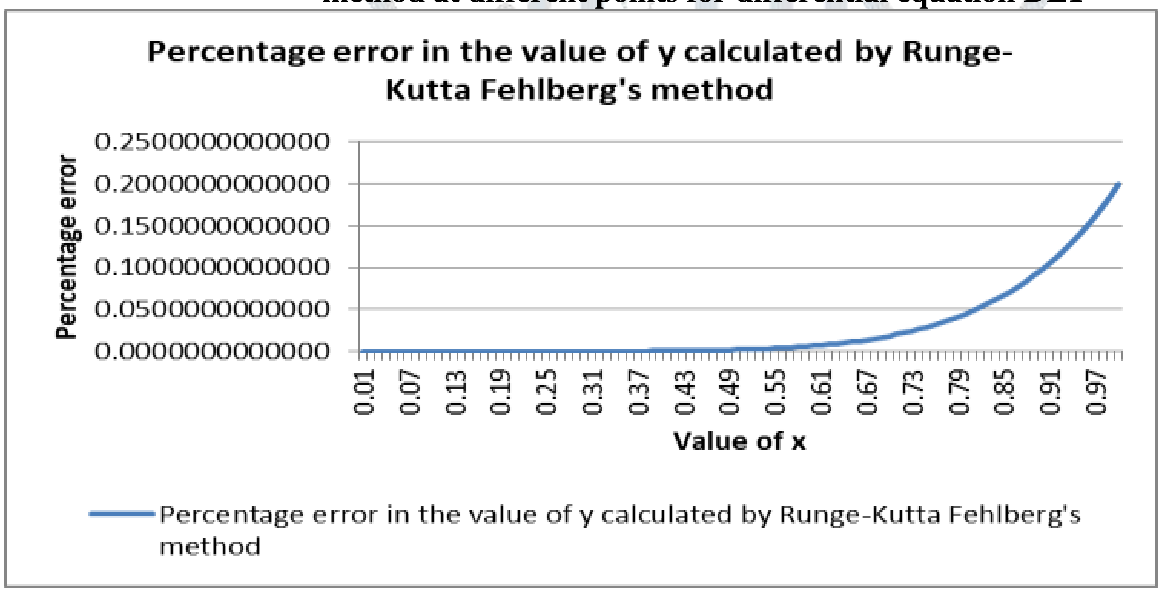
Value of x	Exact value of y for dy/dx = -x, y=1 when x=0	Value of y calculated by Runge-Kutta Fehlberg's method	Error in the value of y calculated by Runge-Kutta Fehlberg's method	Percentage error in the value of y calculated by Runge-Kutta Fehlberg's method
0.33	0.7189237454303	0.7189226583123	0.0000010871179	0.0001512146371
0.34	0.7117703202171	0.7117690153803	0.0000013048368	0.0001833227336
0.35	0.7046880939190	0.7046865438704	0.0000015500486	0.0002199623687
0.36	0.6976763368831	0.6976745006574	0.0000018362257	0.0002631916280
0.37	0.6907343273437	0.6907321793267	0.0000021480170	0.0003109758599
0.38	0.6838614124733	0.6838588874410	0.0000025250322	0.0003692315687
0.39	0.6770568841835	0.6770539329307	0.0000029512529	0.0004358943740
0.40	0.6703200620173	0.6703166425561	0.0000034194612	0.0005101236594
0.41	0.6636502525097	0.6636463116397	0.0000039408700	0.0005938172940
0.42	0.6570468284309	0.6570422798574	0.0000045485736	0.0006922754000
0.43	0.6505091094572	0.6505038815580	0.0000052278992	0.0008036627119
0.44	0.6440364226186	0.6440304449805	0.0000059776381	0.0009281521821
0.45	0.6376281592229	0.6376213148360	0.0000068443869	0.0010734135179
0.46	0.6312836590528	0.6312758712031	0.0000077878497	0.0012336529871
0.47	0.6250022690278	0.6249934301790	0.0000088388488	0.0014142106747
0.48	0.6187833984450	0.6187733964274	0.0000100020175	0.0016164004346
0.49	0.6126264065996	0.6126150974300	0.0000113091696	0.0018460140642
0.50	0.6065306597126	0.6065179344684	0.0000127252443	0.0020980380885
0.51	0.6004955845390	0.6004812810582	0.0000143034809	0.0023819460511
0.52	0.5945205593098	0.5945045255937	0.0000160337161	0.0026969153322
0.53	0.5886049865185	0.5885870408345	0.0000179456839	0.0030488501381
0.54	0.5827482746041	0.5827282431729	0.0000200314312	0.0034374072115
0.55	0.5769498035027	0.5769274872954	0.0000223162073	0.0038679634054
0.56	0.5712090624869	0.5711842516197	0.0000248108672	0.0043435703065
0.57	0.5655254427445	0.5654979232132	0.0000275195313	0.0048661880174
0.58	0.5598983759097	0.5598678980574	0.0000304778523	0.0054434614583
0.59	0.5543272992723	0.5542935748082	0.0000337244641	0.0060838541045
0.60	0.5488116557211	0.5487744430983	0.0000372126228	0.0067805817127
0.61	0.5433508613018	0.5433098544468	0.0000410068550	0.0075470304602
0.62	0.5379444350296	0.5378993398543	0.0000450951753	0.0083828686302
0.63	0.5325918035465	0.5325422694383	0.0000495341082	0.0093005765155
0.64	0.5272924315860	0.5272380992123	0.0000543323737	0.0103040306383
0.65	0.5220457892076	0.5219862831588	0.0000595060487	0.0113986263162
0.66	0.5168513517435	0.5167862838996	0.0000650678439	0.0125892761392
0.67	0.5117085997467	0.5116375504445	0.0000710493021	0.0138847191900
0.68	0.5066169887420	0.5065394766610	0.0000775120810	0.0152999371726
0.69	0.5015760702619	0.5014916608780	0.0000844093840	0.0168288299537
0.70	0.4965853097112	0.4964934860488	0.0000918236624	0.0184910146588
0.71	0.4916442080105	0.4915444325054	0.0000997755051	0.0202942500852
0.72	0.4867522710466	0.4866440202239	0.0001082508227	0.0222394078286
0.73	0.4819090096225	0.4817916376532	0.0001173719693	0.0243556287574

Value of x	Exact value of y for dy/dx = -x, y=1 when x=0	Value of y calculated by Runge-Kutta Fehlberg's method	Error in the value of y calculated by Runge-Kutta Fehlberg's method	Percentage error in the value of y calculated by Runge-Kutta Fehlberg's method
0.74	0.4771139109709	0.4769868497009	0.0001270612700	0.0266312230808
0.75	0.4723665527410	0.4722291175387	0.0001374352023	0.0290950325565
0.76	0.4676664314699	0.4675179604521	0.0001484710178	0.0317472043746
0.77	0.4630130771425	0.4628528234075	0.0001602537350	0.0346110602346
0.78	0.4584060244203	0.4582332518275	0.0001727725928	0.0376898608629
0.79	0.4538448125952	0.4536587130065	0.0001860995886	0.0410051152857
0.80	0.4493289855429	0.4491287118789	0.0002002736640	0.0445717214860
0.81	0.4448580651623	0.4446427619931	0.0002153031692	0.0483981714798
0.82	0.4404316576562	0.4402004031094	0.0002312545468	0.0525063407289
0.83	0.4360492935989	0.4358011394851	0.0002481541138	0.0569096470112
0.84	0.4317105347511	0.4314444433266	0.0002660914245	0.0616365372451
0.85	0.4274149472343	0.4271298984539	0.0002850487804	0.0666913457883
0.86	0.4231621014868	0.4228569726452	0.0003051288416	0.0721068452212
0.87	0.4189515472499	0.4186252019167	0.0003263453332	0.0778957221612
0.88	0.4147829136594	0.4144341597326	0.0003487539268	0.0840810735785
0.89	0.4106557586268	0.4102833158590	0.0003724427678	0.0906946414351
0.90	0.4065696694340	0.4061722450212	0.0003974244128	0.0977506298783
0.91	0.4025242374693	0.4021005074644	0.0004237300049	0.1052681964135
0.92	0.3985190581871	0.3980675609871	0.0004514972000	0.1132937536383
0.93	0.3945537075495	0.3940729747973	0.0004807327522	0.1218421581170
0.94	0.3906278362899	0.3901163476086	0.0005114886813	0.1309401516618
0.95	0.3867410280648	0.3861971560853	0.0005438719795	0.1406295014001
0.96	0.3828928941911	0.3823150140580	0.0005778801331	0.1509247473173
0.97	0.3790830498529	0.3784693869863	0.0006136628666	0.1618808508685
0.98	0.3753111140632	0.3746598917368	0.0006512223264	0.1735153322151
0.99	0.3715767096261	0.3708860670248	0.0006906426013	0.1858681083599
1.00	0.3678794411714	0.3671474364610	0.0007320047104	0.1989795102678

Graph-1: Graph between errors in the value of y calculated by Runge-Kutta Fehlberg's method at different points for differential equation DE1



Graph-2: Graph between percentage errors in the value of y calculated by Runge-Kutta Fehlberg's method at different points for differential equation DE1



Conclusion

Minimum error, maximum error, minimum percentage error and maximum percentage error for each differential equation have been calculated by using Runge-Kutta Fehlberg's method and shown in Table-3. Maximum errors of differential equations indicate that Runge-Kutta Fehlberg's method is at least accurate up to 2 places after the decimal point as the average of maximum errors has been found to be 0.0011957857183. Average of maximum percentage errors in the values of y at different points as calculated by Runge-Kutta Fehlberg's method is 0.0733067638714.

Table-3: Maximum percentage error and maximum error for each differential equation as calculated by Runge-Kutta Fehlberg's method

Differential Equation	Minimum Error in the value of y calculated by Runge-Kutta Fehlberg's method	Maximum Error in the value of y calculated by Runge-Kutta Fehlberg's method	Minimum Percentage Error in the value of y calculated by Runge-Kutta Fehlberg's method	Maximum Percentage Error in the value of y calculated by Runge-Kutta Fehlberg's method
DE1	0.0000000000482	0.0007320047104	0.0000000052723	0.1989795102678
DE2	0.0000000000283	0.0032533696262	0.0000000027061	0.0807694040697
DE3	0.0000000000006	0.0000000285827	0.0000000292230	0.0000098128293
DE4	0.0000000000049	0.0019934672566	0.0000000004930	0.0867689711785
DE5	0.0000000000104	0.0000000584156	0.0000000000104	0.0000061210116
Average	0.0000000000185	0.0011957857183	0.0000000075410	0.0733067638714

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