# CLASSICAL ANALYSIS OF MODERN ALGEBRA TOOLS WITH FIXED POINT THEORY

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*Abstract :* We'll be looking at several kinds of algebraic structures this research paper, the major kinds being fields, rings, and groups, but also minor variants of these structures. We'll start by examining the definitions and looking at some examples. For the time being, we won't prove anything; that will come later when we look at each structure in depth.

***Index Terms* : Fields, rings, and groups**

## 1. INTRODUCTION

### (i) Fields

Informally, a field is a set equipped with four operations—addition, subtraction, multiplication, and division that have the usual properties. (They don't have to have the other operations that R has, like powers, roots, logs, and the myriad functions like sin x.)

**Definition**: A field is a set equipped with two binary operations, one called addition and the other called multiplication, denoted in the usual manner, which are both commutative and associative, both have identity elements.

### (ii) Rings

Rings will have the three operations of addition, subtraction, and multiplication, but don't need division.[1] Most of our rings will have commutative multiplication, but some won't, so we won't require that multiplication be commutative in our definition.

All the rings we'll look at have a multiplicative identity, 1, so we'll include that in the definition. A ring is a set equipped with two binary operations, one called addition and the other called multiplication, denoted in the usual manner, which are both associative, addition is commutative, both have identity elements (the additive identity denoted 0 and the multiplicative identity denoted 1), addition has inverse elements (the inverse of x denoted −x), and multiplication distributes over addition. If, furthermore, multiplication is commutative, then the ring is called a commutative ring.

### (iii) Groups

Unlike fields and rings which have two primary binary operations, groups only have one binary operation.

**Definition:** A group is a set equipped with a binary operation that is associative, has an identity element, and has inverse elements. If, furthermore, multiplication is commutative, then the group is called a commutative group or an Abelian group. Abelian groups can be denoted either additively or multiplicatively, but nonabelian groups are usually denoted multiplicatively. We'll use the term order of the group to indicate how many elements a group G has and denote this order by |G|.

## 1. OPERATION SETS

We're familiar with many operations on the real numbers R. addition, subtraction, multiplication, division, negation, reciprocation, powers, roots, etc. Addition, subtraction, and multiplication are examples of binary operations, that is, functions R × R → R which take two real numbers as their arguments and return another real number.

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Division is almost a binary operation, but since division by 0 is not defined, it's only a partially defined operation. Most of our operations will be defined everywhere, but some won't be.

Negation is a unary operation, that is, a function $R \to R$ which takes one real number as an argument and returns a real number. Reciprocation is a partial unary operation since the reciprocal of zero is not defined. The operations we'll consider are all binary or unary.

Ternary operations can certainly be defined, but useful ones are rare. Some of these operations satisfy familiar identities. For example, addition and multiplication are both commutative; they satisfy the identities $x + y = y + x$ and $xy = yx$.

A binary operation is said to be commutative when the order that the two arguments are applied doesn't matter, that is, interchanging them, or commuting one across the other, doesn't change the result. Subtraction and division, however, are not commutative. Addition and multiplication are also associative binary operations $(x + y) + z = x + (y + z)$ and $(xy)z = x(yz)$

A binary operation is said to be associative when the parentheses can be associated with either the first pair or the second pair when the operation is applied to three arguments and the result is the same. Neither subtraction nor division are associative. Both addition and multiplication also have identity elements $0 + x = x = x + 0$ and $1x = x = x1$.

An identity element, also called a neutral element, for a binary operation is an element in the set that doesn't change the value of other elements when combined with them under the operation. So, 0 is the identity element for addition, and 1 is the identity element for multiplication. Subtraction and division don't have identity elements. Also, there are additive inverses and multiplicative inverses (for nonzero) elements.

That is to say, given any x there is another element, namely $-x$, such that $x + (-x) = 0$, and given any nonzero x there is another element, namely 1 x such that x( 1 x ) = 1.

Thus, a binary operation that has an identity element is said to have inverses if for each element there is an inverse element such that when combined by the operation they yield the identity element for the operation. Addition has inverses, and multiplication has inverses of nonzero elements.

Since $x - 0 = x$ and $x/1 = x$, but not on the left, since usually $0 - x$ 6= x and $1/x$ 6= x.

Finally, there is a particular relation between the operations of addition and multiplication, that of distributive. $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$. Multiplication distributes over addition, that is, when multiplying a sum by x we can distribute the x over the terms of the sum.

## 2. APPLICATION OF MODERN ALGEBRA

Following are the application of modern algebra

(i) Coding Theory
(ii) Graph Theory
(iii) Cryptography
(iv) Error Detection
(v) Queuing Theory
(vi) OR Theory
(vii) Factorization
(viii) Primality
(ix) Traversal Problems
(x) Computability
(xi) Permutation
(xii) Combinatory

## 3. REMARKS

There are numerous useful properties that are logical consequences of the axioms. Generally speaking, the list of axioms should be short, if not minimal, and any properties that can be proved should be proved. Here's a list of several things that can be proved from the axioms. We'll prove a few in paper, you'll prove some as homework, and we'll leave the rest. (They make good questions for quizzes and tests.)

In the following statements, un-quantified statements are meant to be universal with the exception that whenever a variable appears in a denominator, that variable is not to be 0.

Each quaternion a is the sum of a real part a0 and a pure quaternion part a1i + a2j + a3k. Hamilton called the real part a scalar and pure quaternion part a vector. We can interpret a1i + a2j + a3k as a vector a = (a1, a2, a3) in R3 . Addition and subtraction of pure quaternions then are just ordinary vector addition and subtraction.

Hamilton recognized that the product of two vectors (pure quaternions) had both a vector component and a scalar component (the real part). The vector component of the product ab of two pure quaternions Hamilton called the vector product, now often denoted $a \times b$ or $a \wedge b$, and called the cross product or the outer product. The negation of the scalar component Hamilton called the scalar product, now often denoted $a \cdot b$, (a, b), ha, bi, or ha|bi and called the dot product or the inner product. Thus $ab = a \times b - a \cdot b$.

Hamilton's quaternions were very successful in the 19th century in the study of threedimensional geometry. Here's a typical problem from Kelland and Tait's 1873 Introduction to Quaternions. If three mutually perpendicular vectors be drawn from a point to a plane, the sum of the reciprocals of the squares of their lengths is independent of their directions.

## 4. CONCLUSION

Rings have the three operations of addition, subtraction, and multiplication, but don't need division. Most of our rings will have commutative multiplication, but some won't, so we won't require that multiplication be commutative in our definition. We will require that every ring have 1. The formal definition for rings is very similar to that for fields, but we leave out a couple of the requirements.

An integral domain is a commutative ring D in which 0 6= 1 that satisfies one of the two equivalent conditions: it has no zero-divisors, or it satisfies the cancellation law. All the fields and most of the examples of commutative rings we've looked at are integral domains, but Zn is not an integral domain if n is not a prime number. Note that any subring of a field or an integral domain will an integral domain since the subring still won't have any zero-divisors.

Categories are higher order algebraic structures. We'll look at the category of rings in which the objects of the category are all the rings. The purpose of a category is to study the interrelations of its objects, and to do that the category includes morphisms between the objects. In the case of the category of rings, the morphisms are the ring homo-morphisms.

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