Construction of Source Potential due to Three Dimensional Waves in a Two Layered Liquid Media

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Abstract : *A method, essentially based on a suitable superposition of the basic solutions for the water wave potentials along with the application of the inverse Fourier sine transform technique, is used here to find the solution for the three-dimensional problem of incoming waves at the interface of a two layered liquid media. Assuming linear theory, analytical expressions for the velocity potentials are obtained here by ignoring the effect of interfacial tension at the interface of the liquids and also by considering the effect. Various results are recovered as special cases and compared with the known results.*

Key Words : *inviscid liquid, irrotational flow, linear theory, source potential, vertical cliff.*

I. INTRODUCTION

Problem of water waves on a beach which slopes at an angle π/2n with the horizontal, n being any integer, is an important oceanic phenomenon. This problem has attracted the attention of many researchers for a long time. For n=1, i.e. when a vertical wall exists on one side of the ocean, a sloping beach problem reduces to the problem involving a vertical cliff. One such problem is the problem of incoming surface water waves against a vertical cliff. Neglecting the effect of surface tension at the free surface, the solutions of the corresponding two-dimensional as well as three-dimensional problems, were obtained by Stoker [1], [2], long back, for deep water case. He used a powerful, though, complicated method essentially based on the theory of analytic functions of complex variables to solve the problems. The effect of surface tension at the free surface for the two-dimensional vertical cliff problem has been studied by Packham [3]. The analysis of Packham [3] is essentially based on a reduction procedure along with the application of the Fourier sine transform technique (cf. [4]). Since then, few attempts have been made to study this class of water wave problems and few of its generalization by employing different mathematical techniques (cf. [5]- [12]). Although problem of incoming waves progressing towards a single liquid, have been considered by several investigators in the past and recent years, however, the problem of waves at the interface of two superposed liquids in presence of a vertical cliff is rather limited. To the authors' knowledge the first problem along this direction has been considered by Kundu [13], wherein a method essentially based on a simple reduction procedure was employed to find the solution of the problem.

Present study is concerned with the three-dimensional problem of incoming waves, at the interface of two liquids, progressing towards a vertical cliff of infinite length. In this analysis, no reflection of waves by the cliff is assumed. Generally, the cliff bound wave carries certain energy with it and is totally reflected back, if there is no mechanism to absorb (or dissipate) the incoming energy in an inviscid fluid system (cf. [6]) with a rigid cliff. For the present investigation, the assumption of no reflection of waves by the cliff can be justified by introducing a source/sink type behavior in the potential functions at the origin, i.e., where the interface of two liquids meet the cliff, which requires logarithmic singularity in the potential functions at the origin (cf. [14]). However, in the presence of surface tension, this requirement of logarithmic singularity of the potential functions at the origin is not necessary, since the wave amplitude remains finite there (cf. [3]).

In this paper we have considered the three-dimensional problem of incoming waves at the surface of separation of two immiscible liquids against a rigid vertical cliff, where the lower liquid extends infinitely downwards and the upper liquid extends infinitely upwards. A method essentially based on a suitable superposition of the basic solutions for the water wave potentials (cf. [15]) and the inverse Fourier sine transform technique (cf. [4]) is used to find the solution of the problem under consideration. The solution is also obtained here by assuming the effect of interfacial tension at the interface of the liquids. Various results are recovered as special cases of the general problem considered here and identified with the known results.

II. FORMULATION AND SOLUTION OF THE PROBLEM

Consider the three-dimensional irrotational motion of two inviscid, immiscible, homogeneous liquid of densities ρ_1 and ρ_2 ($<\rho_1$), for the lower and upper liquid, respectively. Cartesian co-ordinate system is chosen in which the y-axis is taken to be vertically downwards into the lower liquid, so that the plane y=0, x>0 is the undisturbed interface, and the cliff is given by x=0, $-\infty<y<\infty$. Lower and upper liquid occupies the regions x>0, y>0, $-\infty<z<\infty$ and x>0, y<0, $-\infty<z<\infty$ respectively. The origin is taken at the point on the line of intersection where the mean interface and the cliff meets.

Since the motion is assumed to be irrotational, there exist velocity potentials $\Phi_i(x,y,z,t)$, (i=1, 2; i=1 is used for the lower liquid, while i=2 is used for the upper liquid throughout the paper), which represent progressive waves moving towards the shore line (i.e. the z-axis), such that the wave crests at a large distance from the shore tend to straight lines which make an arbitrary angle α with the shore line.

Thus for periodic motion, we can assume

$$\left.\begin{aligned}\Phi_1(x,y,z,t) &= Re[\,\phi_1(x,y)exp\{-i(\sigma t+\nu z)\}]\\ \Phi_2(x,y,z,t) &= Re[\,\phi_2(x,y)exp\{-i(\sigma t+\nu z)\}]\end{aligned}\right\} \qquad (2.1)$$

where $\nu = L\sin\alpha$, L is defined, later on, by (2.8), and σ is the circular frequency.
Assuming linear theory, the potential functions $\phi_1(x,y)$ and $\phi_2(x,y)$ satisfy:
(i) Two-dimensional modified Helmholtz's equations :

$$\left.\begin{aligned}(\nabla^2-\nu^2)\phi_1=0\\(\nabla^2-\nu^2)\phi_2=0\end{aligned}\right\}\text{ in the respective region of liquid} \qquad (2.2)$$

where ∇^2 is the two dimensional Laplacian.

(ii) Linearized form of the interface conditions :

$$\left.\begin{aligned}\phi_{1_y}&=\phi_{2_y}\\K\phi_1+\phi_{1_y}&=s\left(K\phi_2+\phi_{2_y}\right)\end{aligned}\right\}\text{ on } y=0, x>0 \qquad (2.3)$$

where $K=\sigma^2/g$, the wave number, g being the acceleration due to gravity and $s=\rho_2/\rho_1$.

(iii) The condition of vanishing of the normal component of velocity at the vertical cliff:

$$\left.\begin{aligned}\phi_{1_x}=0,\quad for\ y>0\\\phi_{2_x}=0,\quad for\ y<0\end{aligned}\right\}\text{ on } x=0. \qquad (2.4)$$

(iv) The condition of no motion at infinite depth and height :

$$\left.\begin{aligned}\nabla\phi_1\to 0\quad as\ y\to\infty\\\nabla\phi_2\to 0\quad as\ y\to-\infty\end{aligned}\right\} \qquad (2.5)$$

Finally, no reflection of waves by the cliff is assumed. Thus in the absence of surface tension, a source/sink type behavior of the potential functions at the shore line is necessary, so that the wave amplitude becomes infinite at the origin and as such (cf. [14])

$$\phi_1,\phi_2 \sim \ln r \quad as\ r=(x^2+y^2)^{1/2}\to 0. \qquad (2.6)$$

Our aim is to obtain $\phi_1(x,y)$ and $\phi_2(x,y)$ satisfying (2.2) - (2.5), together with the condition that they behave at infinity like progressive waves moving towards the cliff, so that following conditions (2.3) and (2.5), we have

$$\left.\begin{aligned}\phi_1&\sim\exp(-Ly-i\mu x)\\\phi_2&\sim-\exp(Ly-i\mu x)\end{aligned}\right\}\ as\ x\to\infty, \qquad (2.7)$$

where $\pi = L\,cos\alpha$ and $L=\frac{1+s}{1-s}K$.

In view of (2.7), ϕ_1,ϕ_2 can be represented by superposing the basic solutions $\exp(-Ly-i\mu x)$, $(kcos\ ky-L\sin ky)$ $\exp(-k_1x)$ and $-\exp(Ly-i\mu x)$, $(kcos\ ky+L\sin ky)\exp(-k_1x)$ respectively (cf. [15]) where $k_1=(k^2+\upsilon^2)^{1/2}$, so that we can assume

$$\left.\begin{aligned}\phi_1(x,y)&=\exp(-Ly-i\mu x)+\int_0^\infty A(k)(kcos\ ky-L\sin ky)\exp(-k_1x)\,dk\\\phi_2(x,y)&=\exp(Ly-i\mu x)-\int_0^\infty B(k)(kcos\ ky+L\sin ky)\exp(-k_1x)\,dk\end{aligned}\right\}\ x>0, \qquad (2.9)$$

where A(k), B(k) are to be determined.

Exploiting the condition (2.4) into (2.9), we find that

$$\left.\begin{aligned}\int_0^\infty k_1A(k)(kcos\ ky-L\sin ky)dk&=-i\mu\exp(-Ly), & y>0\\\int_0^\infty k_1B(k)(kcos\ ky+L\sin ky)dk&=-i\mu\exp(-Ly), & y<0\end{aligned}\right\} \qquad (2.10)$$

It can be easily shown that the two integral equations, given by (2.10), can be reduced to the ordinary differential equations

$$\left.\begin{aligned}\left(\frac{d}{dy}-L\right)f_1(y)&=-i\mu\exp(-Ly), & y>0\\\left(\frac{d}{dy}+L\right)f_2(y)&=-i\mu\exp(Ly), & y<0\end{aligned}\right\} \qquad (2.11)$$

where

$$\left.\begin{aligned} f_1(y) &= \int_0^\infty k_1 A(k) \sin ky \, dk, \qquad y > 0 \\ f_2(y) &= \int_0^\infty k_1 B(k) \sin ky \, dk, \qquad y < 0. \end{aligned}\right\} \tag{2.12}$$

Bounded solutions of the ordinary differential equations given by (2.11) are

$$\left.\begin{aligned} f_1(y) &= \frac{i \cos\alpha}{2} \exp(-Ly) \\ f_2(y) &= -\frac{i \cos\alpha}{2} \exp(Ly). \end{aligned}\right\} \tag{2.13}$$

Using these solutions in (2.12) and invoking inverse Fourier sine transform (cf. [4]), the solutions of the integral equations given by (2.10) are obtained as

$$A(k) = B(k) = \frac{ik \cos\alpha}{\pi k_1 (k^2 + L^2)}$$

so that from (2.9) we find

$$\left.\begin{aligned} \phi_1(x,y) &= \exp(-Ly - i\mu x) + \frac{i \cos\alpha}{\pi} \int_0^\infty \frac{k(k\cos ky - L \sin ky)}{k_1(k^2+L^2)} \exp(-k_1 x) \, dk \\ \phi_2(x,y) &= -\exp(Ly - i\mu x) - \frac{i \cos\alpha}{\pi} \int_0^\infty \frac{k(k\cos ky + L \sin ky)}{k_1(k^2+L^2)} \exp(-k_1 x) \, dk. \end{aligned}\right\} \tag{2.14}$$

It is to be noted here that the integrals in (2.14) have a logarithmic singularity at the origin (cf. [14]) and ϕ_1, ϕ_2 satisfy the conditions (2.3) to (2.7). Thus, the required velocity potentials can be found from (2.1), which are given by

$$\left.\begin{aligned} \Phi_1(x,y,z,t) &= \exp(-Ly)\cos(\mu x + \sigma t + \nu z) + \frac{\cos\alpha}{\pi} \sin(\sigma t + \nu z) \\ &\quad \times \int_0^\infty \frac{k(k\cos ky - L \sin ky)}{k_1(k^2+L^2)} \exp(-k_1 x) \, dk \\ \Phi_2(x,y,z,t) &= -\exp(Ly)\cos(\mu x + \sigma t + \nu z) - \frac{\cos\alpha}{\pi} \sin(\sigma t + \nu z) \\ &\quad \times \int_0^\infty \frac{k(k\cos ky + L \sin ky)}{k_1(k^2+L^2)} \exp(-k_1 x) \, dk \end{aligned}\right\} \tag{2.15}$$

It is interesting to note here that the solution for the corresponding two-dimensional problem can be deduced, directly, by the substitution of $\alpha = 0$ in (2.15).

If the density of the upper liquid ρ_2 be made equal to zero (then $s = 0$), so that $L = K$ in the expression for Φ_1 (the velocity potential for the lower liquid), given by (2.15), the solution for the corresponding three-dimensional problem in a liquid can be obtained as

$$\begin{aligned} \Phi_1(x,y,z,t) &= \exp(-Ky)\cos(\mu x + \sigma t + \nu z) + \frac{\cos\alpha}{\pi} \sin(\sigma t + \nu z) \\ &\quad \times \int_0^\infty \frac{k(k \cos ky - K \sin ky)}{k_1(k^2+K^2)} \exp(-k_1 x) \, dk. \end{aligned} \tag{2.16}$$

The solution for the corresponding two-dimensional problem in a single liquid can be deduced, directly, from the above expression by the substitution of $\alpha = 0$, which is given by

$$\Phi_1(x,y,t) = \exp(-Ky)\cos(Kx + \sigma t) + \frac{\sin\sigma t}{\pi} \int_0^\infty \frac{(k\cos ky - K \sin ky)}{(k^2+K^2)} \exp(-kx) \, dk. \tag{2.17}$$

The results given by (2.16) and (2.17) were obtained by Stoker [1], [2], long back, using a different mathematical technique, for a single liquid.

III. Effect of Interfacial Tension

If the effect of interfacial tension is considered, the problem already described in section 2 can be reformulated as: find the solution in the form

$$\left.\begin{array}{l}\Phi_1(x,y,z,t) = Re[\varphi_1(x,y)exp\{-i(\sigma t+\nu_0 z)\}] \\ \Phi_2(x,y,z,t) = Re[\varphi_2(x,y)exp\{-i(\sigma t+\nu_0 z)\}]\end{array}\right\} \quad (3.1)$$

where $\nu_0 = L_0 \sin\alpha$, L_0 is defined, later on, by (3.6).

In this case $\phi_1(x,y)$, $\phi_2(x,y)$ must satisfy:

$$\left.\begin{array}{l}(\nabla^2-{\nu_0}^2)\phi_1 = 0 \\ (\nabla^2-{\nu_0}^2)\phi_2 = 0\end{array}\right\} \quad \text{in the respective domain of liquid.} \quad (3.2)$$

The linearized interface conditions:

$$\left.\begin{array}{l}\phi_{1_y} = \phi_{2_y} \\ K\phi_1+\phi_{1_y} - s\left(K\phi_2+\phi_{2_y}\right) + M\begin{Bmatrix}\phi_{1_{yyy}} \\ \phi_{2_{yyy}}\end{Bmatrix} = 0\end{array}\right\} \text{ on } y=0, x>0\,, \quad (3.3)$$

where $M = \frac{T}{\rho_1 g}$, T being the co-efficient of interfacial tension, together with the conditions given by (2.4) and (2.5).

Since in the presence of surface tension, the wave amplitude remains finite at the origin (cf.[3]), instead of (2.6), in this case we have

$$\phi_1, \phi_2 \text{ remains finite as } r\to 0, \quad (3.4)$$

as long as the interfacial tension effect $T > 0$.

As $\phi_1(x,y)$, $\phi_2(x,y)$ behave like incoming progressive waves at infinity, we must have

$$\left.\begin{array}{l}\phi_1 \sim \exp(-L_0 y - i\mu_0 x) \\ \phi_2 \sim -\exp(L_0 y - i\mu_0 x)\end{array}\right\} \quad as\ x\to\infty, \quad (3.5)$$

where $\mu_0 = L_0 \cos\alpha$ and L_0 is the unique real positive root of the cubic equation

$$k\left(1+M^{'}k^2\right) - L = 0, \quad (3.6)$$

with $M^{'} = \frac{M}{1-s}$.

The equation (3.6) has also a pair of complex conjugate roots, say ω_1, ω_2, whose real part is negative (cf. [16]).

In view of (3.5), we find that for $x>0$, $\phi_1(x,y)$, $\phi_2(x,y)$ can be expressed as a superposition of the basic solutions $\exp(-L_0 y - i\mu_0 x)$, $\{k\left(1-M^{'}k^2\right)\cos ky - L\sin ky\}\exp(-k_2 x)$ and $-\exp(L_0 y - i\mu_0 x)$, $-\{k\left(1-M^{'}k^2\right)\cos ky + L\sin ky\}\exp(-k_2 x)$ respectively (cf. [15]) where $k_2 = (k^2+v_0^2)^{1/2}$, so that for x > 0

$$\left.\begin{array}{l}\phi_1(x,y) = \exp(-L_0 y - i\mu_0 x) + \int_0^\infty C(k)\{k\left(1-M^{'}k^2\right)\cos ky - L\sin ky\}\exp(-k_2 x)\,dk \\ \phi_2(x,y) = -\exp(L_0 y - i\mu_0 x) - \int_0^\infty D(k)\{k\left(1-M^{'}k^2\right)\cos ky + L\sin ky\}\exp(-k_2 x)\,dk\end{array}\right\} \quad (3.7)$$

where $C(k)$ and $D(k)$ are unknowns.

Using the condition (2.4) into (3.7), we find that

$$\left.\begin{array}{ll}\int_0^\infty k_2 C(k)\{k\left(1-M^{'}k^2\right)\cos ky - L\sin ky\}dk = -i\mu_0\exp(-L_0 y)\,, & y>0 \\ \int_0^\infty k_2 D(k)\{k\left(1-M^{'}k^2\right)\cos ky + L\sin ky\}dk = -i\mu_0\exp(L_0 y)\,, & y<0\end{array}\right\} \quad (3.8)$$

The above integral equations can be reduced to the ordinary differential equations given by

$$\left.\begin{array}{ll}\left(M' \dfrac{d^3}{dy^3} + \dfrac{d}{dy} - L\right) g_1(y) = -i\mu_0 \exp(-L_0 y), & y > 0 \\ \left(M' \dfrac{d^3}{dy^3} + \dfrac{d}{dy} + L\right) g_2(y) = -i\mu_0 \exp(L_0 y), & y < 0\end{array}\right\} \tag{3.9}$$

where

$$\left.\begin{array}{ll} g_1(y) = \displaystyle\int_0^\infty k_2 C(k) \sin ky \, dk, & y > 0 \\ g_2(y) = \displaystyle\int_0^\infty k_2 D(k) \sin ky \, dk, & y < 0. \end{array}\right\} \tag{3.10}$$

Bounded solutions of the ordinary differential equations (3.9) are given by

$$\left.\begin{array}{l} g_1(y) = \dfrac{i\,\mu_0}{2L}\{\exp(-L_0 y) + A_1 \exp(\omega_1 y) + B_1 \exp(\omega_2 y)\} \\ g_2(y) = -\dfrac{i\,\mu_0}{2L}\{\exp(L_0 y) + A_2 \exp(-\omega_1 y) + B_2 \exp(-\omega_2 y)\} \end{array}\right\} \tag{3.11}$$

where A_1, B_1, A_2, B_2 are arbitrary constants.

Utilizing these solutions in (3.10) and exploiting inverse Fourier sine transform technique (cf.[4]), we find the solutions of the integral equations (3.8) as

$$C(k) = \frac{i\,\mu_0 k}{\pi L k_2}\left[\frac{1}{k^2 + L_0^2} + \frac{A_1}{k^2 + \omega_1^2} + \frac{B_1}{k^2 + \omega_2^2}\right], \qquad D(k) = \frac{i\,\mu_0 k}{\pi L k_2}\left[\frac{1}{k^2 + L_0^2} + \frac{A_2}{k^2 + \omega_1^2} + \frac{B_2}{k^2 + \omega_2^2}\right].$$

Since L_0, ω_1, ω_2 are the roots of the cubic equation (3.6), it can be easily be shown that

$$M'^2 (k^2 + L_0^2)(k^2 + \omega_1^2)(k^2 + \omega_2^2) = k^2 \left(1 - M' k^2\right)^2 + L^2,$$

so that the above expressions for $C(k)$and $D(k)$ reduce to the form

$$\left.\begin{array}{l} C(k) = \dfrac{i\,\mu_0 k M'^2 (C_1 k^4 + D_1 k^2 + E_1)}{\pi L k_2 \{k^2 (1 - M' k^2)^2 + L^2\}} \\ D(k) = \dfrac{i\,\mu_0 k M'^2 (C_2 k^4 + D_2 k^2 + E_2)}{\pi L k_2 \{k^2 (1 - M' k^2)^2 + L^2\}} \end{array}\right\} \tag{3.12}$$

where

$$\left.\begin{array}{l} C_j = 1 + A_j + B_j \\ D_j = \omega_1^2 + \omega_2^2 + A_j (L_0^2 + \omega_2^2) + B_j (L_0^2 + \omega_1^2) \\ E_j = \omega_1^2 \omega_2^2 + L_0^2 \left(A_j \omega_2^2 + B_j \omega_1^2\right). \end{array}\right\} \quad (j = 1,2) \tag{3.13}$$

To satisfy the boundedness condition of $\phi_1(x, y)$ and $\phi_2(x, y)$ at the origin, we must have $C_j = 0, D_j = 0$ $(j = 1,2)$. Thus, after some elementary manipulation we obtain

$$A_j = \frac{2\left(\frac{1}{M'} + L_0^2\right) + \omega_1^2}{\omega_2^2 - \omega_1^2}, \qquad B_j = -\frac{2\left(\frac{1}{M'} + L_0^2\right) + \omega_2^2}{\omega_2^2 - \omega_1^2}.$$

Therefore from (3.13) we obtain

$$E_j = \frac{L\left(1 + 3M' L_0^2\right)}{M'^2 L_0},$$

so that, from (3.12) we obtain finally

$$C(k) = D(k) = \frac{ik\left(1 + 3M' L_0^2\right) \cos\alpha}{\pi k_2 \{k^2 (1 - M' k^2)^2 + L^2\}}.$$

Thus from (3.7), the potential functions $\phi_1(x, y)$ and $\phi_2(x, y)$ satisfying (3.3), (2.4), (2.5) and (3.5) are given by

$$\left.\begin{aligned}\phi_1(x,y) &= \exp(-L_0 y - i\mu_0 x) + \frac{i\left(1+3M'L_0^2\right)\cos\alpha}{\pi}\\ &\times\int_0^\infty \frac{k\{k\left(1-M'k^2\right)\cos ky - L\sin ky\}}{k_2\{k^2(1-M'k^2)^2+L^2\}}\exp(-k_2x)\,dk\\ \phi_2(x,y) &= -\exp(L_0 y - i\mu_0 x) - \frac{i\left(1+3M'L_0^2\right)\cos\alpha}{\pi}\\ &\times\int_0^\infty \frac{k\{k\left(1-M'k^2\right)\cos ky + L\sin ky\}}{k_2\{k^2(1-M'k^2)^2+L^2\}}\exp(-k_2x)\,dk.\end{aligned}\right\} \quad (3.14)$$

We may note here that so long as $T>0$, ϕ_1,ϕ_2 remain finite as $r\to 0$ (cf. [3]). Hence the explicit form of the velocity potentials become

$$\left.\begin{aligned}\Phi_1(x,y,z,t) &= \exp(-L_0 y)\cos(\mu_0 x+\sigma t+\nu_0 z) + \frac{\left(1+3M'L_0^2\right)\cos\alpha}{\pi}\sin(\sigma t+\nu_0 z)\\ &\times\int_0^\infty \frac{k\{k\left(1-M'k^2\right)\cos ky - L\sin ky\}}{k_2\{k^2(1-M'k^2)^2+L^2\}}\exp(-k_2x)\,dk\\ \Phi_2(x,y,z,t) &= -\exp(L_0 y)\cos(\mu_0 x+\sigma t+\nu_0 z) - \frac{\left(1+3M'L_0^2\right)\cos\alpha}{\pi}\sin(\sigma t+\nu_0 z)\\ &\times\int_0^\infty \frac{k\{k\left(1-M'k^2\right)\cos ky + L\sin ky\}}{k_2\{k^2(1-M'k^2)^2+L^2\}}\exp(-k_2x)\,dk\,.\end{aligned}\right\} \quad (3.15)$$

It may be noted here that simply by assuming $\alpha=0$ in (3.15), the solution for the corresponding two-dimensional problem can be obtained (cf. [13]).Neglecting the interfacial tension effect at the interface of the two liquid (i.e. when $T=0$), $M'=0$ so that $L_0=L$, the expressions for Φ_1,Φ_2 given by (3.15) reduce to that given by (2.15). Also in the absence of upper liquid $s=0$, so that $M'=M$ the expression for Φ_1 given by (3.15) reduces to the velocity potential of a three-dimensional incoming wave in a deep sea against a vertical cliff in the presence of surface tension at the free surface. The explicit form of the velocity potential, in that case, becomes

$$\begin{aligned}\Phi_1(x,y,z,t) &= \exp(-L_0 y)\cos(\mu_0 x+\sigma t+\nu_0 z) + \frac{\left(1+3ML_0^2\right)\cos\alpha}{\pi}\sin(\sigma t+\nu_0 z)\\ &\times\int_0^\infty \frac{k\{k(1-Mk^2)\cos ky - K\sin ky\}}{k_2\{k^2(1-Mk^2)^2+K^2\}}\exp(-k_2x)\,dk\end{aligned} \quad (3.16)$$

which is in complete agreement with the result obtained by Mandal and Kundu [8] in connection with the corresponding problem for a single liquid. In addition, if we assume $\alpha=0$ in the expression for the velocity potential given by (3.16), we obtain

$$\begin{aligned}\Phi_1(x,y,z,t) &= \exp(-L_0 y)\cos(L_0 x+\sigma t) + \frac{\left(1+3ML_0^2\right)}{\pi}\sin\sigma t\\ &\times\int_0^\infty \frac{k\{k(1-Mk^2)\cos ky - K\sin ky\}}{k_2\{k^2(1-Mk^2)^2+K^2\}}\exp(-k_2x)\,dk\end{aligned}$$

which was derived by Packham [3] using a different approach based on a reduction procedure, in connection with the corresponding two-dimensional problem in a single liquid.

IV. Discussion

A relatively simple approach is described here to find the solution of three-dimensional problem of incoming waves progressing at the interface of two liquids against a rigid vertical cliff. Analytical expressions for the velocity potentials in each of the liquids are obtained assuming the lower and the upper liquid to be of infinite depth and height respectively. Using linear theory, the solution is obtained by exploiting a method based on a suitable superposition of the basic solutions for the water wave potential and the inverse Fourier sine transform technique. Reflection of waves at the shore line is not allowed, which requires logarithmic singularities in the potential functions at the cliff (in the absence of surface tension). It may be noted that nonlinear effects and mixing become important near the cliff due to these singularities. Mixing near the cliff is likely to produce an intrustive flow that blurs the interface. However, as the problem is formulated within the frame work of linearized theory, and as the liquids are assumed to be immiscible, these effects are not considered here. The solutions for the potential functions are obtained here by ignoring the effect of interfacial tension first and then by considering the effect of interfacial tension at the interface of two liquids. Various results are recovered as special cases of the general problem considered here. The most important feature of the method described in this paper is that simply by the substitution of α (the angle of incidence) to be equal to zero, results for the corresponding two-dimensional problem can be recovered.

REFERENCES

[1] Stoker, J. J. 1947. Surface waves in water of variable depth. Quart. Appl. Math., 5: 1-54.
[2] Stoker, J. J. 1957. Water waves, Interscience, New York.
[3] Packham, B. A. 1968. Capillary-gravity waves against a vertical cliff. Proc. Camb. Phil. Soc., 64: 827-832.
[4] Sneddon, I. N. 1972. The Use of Integral Transforms, McGraw-Hill, New York.
[5] Rhodes-Robinson, P.F. 1982. Note on the reflection of water waves at a wall in the presence of surface tension, Math. Proc. Camb. Phil. Soc.., 92: 369-373.
[6] Debnath, L. and Basu, U. 1984. Capillary gravity waves against a vertical cliff. Indian J. Maths., 26: 49-56.
[7] Chakrabarti, A. 1988. Capillary gravity waves against a corrugated vertical cliff. Appl. Sci. Res., 45: 303-317.
[8] Mandal, B. N. and Kundu, P.K. 1989. Incoming water waves against a vertical cliff in an ocean. Proc. Indian. Natn. Sci. Acad., 55A: 643-654.
[9] Chakrabarti, A. 1992. Obliquely incident water waves against a vertical cliff. Appl. Math. Lett., 5(1): 13-17.
[10] Chakrabarti, A. and Sahoo, T. 1997. On the incoming water waves against a vertical cliff. Proc. Indian Acad. Sci. (Math. Sci.)., 107: 89-93.
[11] Chakrabarti, A. and Ahluwalia, D. S. May 2000. Surface water waves against a vertical wall in the presence of an ice cover. Proceeding of the 3rd international conference on the advances in Fluid Mechanics-III was held in Canada, 605-615.
[12] Chakrabarti, A. 2003. Incoming water waves against an overhanging cliff. Coastal Engineering VI - 6th international conference on computer modeling and experimental measurement of seas and coastal regions was held in Cadiz., 313-322.
[13] Kundu, P. K. 1989. Incoming water waves against a vertical cliff in a two-fluid medium. Indian J. Pure appl. Math., 20(3): 292-295.
[14] Yu, Y. S. and Ursell, F. 1961. Surface waves generated by an oscillating circular cylinder on water of finite depth: theory and experiment. J. Fluid Mech., 11 : 529-551.
[15] Mandal, B. N. and Chakrabarti, A. 2000. Water wave scattering by barriers. WIT press. Southampton.
[16] Rhodes-Robinson, P. F. 1971. On the forced surface waves due to a vertical wave maker in the presence of surface tension, Proc. Camb. Phil. Soc., 70 : 323-337.