LIE SYMMETRIES AND REDUCTIONS OF (2+1) DIMENSIONAL MODIFIED EQUAL WIDTH WAVE EQUATION WITH DAMPING TERM

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ABSTRACT

In this paper, we are consider a (2+1)-dimensional Modified Equal Width Wave equation with damping term is $u_t + u^3 u_x + u - e^{\sigma t}(u_{xxt} + u_{yyt}) = 0$ subjected to Lies classical method. Classification of its symmetry algebra into one - and two-dimensional subalgebras is carried out in order to facilitate its reduction systematically to (1+1)-dimensional PDE and then to first order ODE.

KEYWORDS - Nonlinear PDE, Lie's Classical Method, Lie's Algebra, Symmetry group.

I. INTRODUCTION

A simple model equation is the Korteweg-de Vries (KdV) equation [10]

$$V_t + 6VV_x + V_{xxx} = 0 \tag{1}$$

Which describe the long waves in shallow water. Its modified version is,

$$u_t - au^2 u_x + u_{xxx} = 0 \tag{2}$$

and again there is Miura transformation [12]

$$V = u^2 + u_x \tag{3}$$

between the KdV equation (1) and its modified version (2). In 2002, Liu and Yang [9] studied the bifurcation properties of generalized KdV equation (GKdVE)

$$u_t + au^n u_x + u_{xxx} = 0 \quad a \in \mathbb{R}\ ,\ \ n \in \mathbb{Z}^+ \tag{4}$$

Gungor and Winternitz [12] transformed the Generalized Kadomtsev-Petviashvili Equation (GKPE)

$$(u_t + p(t)uu_x + q(t)u_{xxx})_x + \sigma(y,t)u_{yy} + a(y,t)u_y + b(y,t)u_{xy} + c(y,t)u_{xx} + e(y,t)u_x + f(y,t)u + h(y,t) = 0 \tag{5}$$

to its canonical form and established conditions on the coefficient functions under which (5) has an infinite dimensional symmetry group having a Kac-Moody-Virasoro structure. In [1-4], they carried out the symmetry analysis of Variable Coefficient Kadomtsev Petviashvili Equation (VCKP) in the form,

$$(u_t + f(x,y,t)uu_x + g(x,y,t)u_{xxx})_x + h(x,y,t)u_y = 0$$

Burgers' equation $u_t + uu_x = \gamma u_{xx}$, is the simplest second order NLPDE which balances the e_ect of nonlinear convection and the linear diffusion. In this paper, we discuss the symmetry reductions of the (2+1)-dimensional modified Equal Width Wave equation with damping term as,

$$u_t + u^3 u_x + u - e^{\sigma t}(u_{xxt} + u_{yyt}) = 0 \tag{6}$$

Our intention is to show that equation (6) admits a three-dimensional symmetry group and determine the corresponding Lie algebra, classify the one-and two-dimensional subalgebras of the symmetry algebra of (6) in order to reduce (6) to (1+1)-dimensional PDEs and then to ODEs. We shall establish that the symmetry generators form a closed Lie algebra and this allowed us to use the recent method due to Ahmad, Bokhari, Kara and Zaman [12] to successively reduce (6) to (1+1)-dimensional PDEs and ODEs with the help of two-dimensional abelian and non-abelian solvable subalgebras. In this work, organized as follows: First, we determine the symmetry group of (6) and write down the associated Lie algebra. secondly, we consider all one-dimensional subalgebras and obtain the corresponding reductions to (1+1)-dimensional PDEs. Next, we show that the generators form a closed Lie algebra and use this fact to reduce (6)successively to (1+1)-dimensional PDEs and ODEs. Finally, we summarises the conclusions of the present work.

II. Symmetries and classifications of lie algebra for $\mathbf{u_t + u^3 u_x + u - e^{\sigma t}(u_{xxt} + u_{yyt}) = 0}$

In order to derive the symmetry generators of Eqn. (1) and obtain the closed form solutions for all f(u), we consider one parameter Lie point transformation that leaves (1) invariant. This transformation is given by

$$\tilde{x}^i = x^i + \in \xi^i(x,y,t,u) + o(\in^2) \quad i = 1, \ldots..4,$$

Where $\xi^i = \frac{\partial \tilde{x}^i}{\partial \in}\Big|_{\in=0}$ defines the symmetry generator associated with (2) given by

$$V = \xi \frac{\partial}{\partial x^i} + \eta \frac{\partial}{\partial x^i} + \zeta \frac{\partial}{\partial x^i} + \varphi \frac{\partial}{\partial x^i}. \tag{7}$$

In order to determine four components ξ^i , we prolong V to third order. we prolong V to third order

$$V^{(3)} = v + \varphi^x \frac{\partial}{\partial u_x} + \varphi^y \frac{\partial}{\partial u_y} + \varphi^t \frac{\partial}{\partial u_t} + \varphi^{xx} \frac{\partial}{\partial u_{xx}} + \varphi^{xy} \frac{\partial}{\partial u_{xy}} + \varphi^{xt} \frac{\partial}{\partial u_{xt}} + \varphi^{yy} \frac{\partial}{\partial u_{yy}} + \varphi^{yt} \frac{\partial}{\partial u_{yt}} + \varphi^{tt} \frac{\partial}{\partial u_{tt}} + \varphi^{xxx} \frac{\partial}{\partial u_{xxx}} + \varphi^{xxy} \frac{\partial}{\partial u_{xxy}} + \varphi^{xxt} \frac{\partial}{\partial u_{xxt}} + \varphi^{xyy} \frac{\partial}{\partial u_{xyy}} + \varphi^{yyt} \frac{\partial}{\partial u_{yyt}} + \varphi^{ttt} \frac{\partial}{\partial u_{ttt}} \tag{8}$$

In above expression every coefficient of the prolonged generator is a functions of (x,y,t,u) and can be determined by the formulae

$$\varphi^i = D_i(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,I} + \eta u_{y,i} + \tau u_{t,i}\,,$$

$$\varphi^{ij} = D_i D_j(\varphi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij}. \tag{9}$$

Where D_i represents total derivative and subscripts of *u* derivative with respect to the respective coordinates. To proceed with reductions of (1) we now use symmetry criterion for partial differential equations. For heat equation this criterion is expressed by the formula

$$V^3[u_t + u^3 u_x + u - e^{\sigma t}(u_{xxt} + u_{yyt})] = 0$$

Whenever

$$u^3 u_x + u - e^{\sigma t}(u_{xxt} + u_{yyt}) = 0 \tag{10}$$

Using the symmetry criterion with Eqn.(4) in mind immediately yields.

$$\varphi^3 \varphi^x + \varphi - e^{\sigma t}(\varphi^{xxt} + \varphi^{yyt}) = 0 \tag{11}$$

At this stage we calculate expression for $\varphi^3\varphi^x, \varphi, \varphi^{xxt}, \varphi^{yyt}$ using (5)-(6), substitute them in (6) and then compare coefficients of various monomials in derivative of u. this yields the following system of over-determined partial differential equations:

$\xi_u = 0$,

$\eta_u = 0$,

$\tau_u = 0,$

$\varphi_{uu} = 0$

$\xi_t = 0,$

$\eta_t = 0,$

$\varphi_{tu} = 0,$

$\tau_y = 0,$

$\tau_x = 0,$

$\varphi + u\tau_t + \varphi_t - u\varphi_u + e^{\sigma t}\varphi_{xxt} + e^{\sigma t}u\varphi_{xxu} - e^{\sigma t}\varphi_{yyt} + e^{\sigma t}u\varphi_{yyu} = 0,$

$3u^3\varphi - u^3\xi_x + u^3\tau_t - 2e^{\sigma t}\varphi_{xtu} + e^{\sigma t}u^3\varphi_{xxu} + e^{\sigma t}u^3\varphi_{yyu} = 0,$

$-2\xi_x + e^{\sigma t}\varphi_{xxu} + e^{\sigma t}u^3\varphi_{yyu} = 0,$

$-2\eta_y + e^{\sigma t}\varphi_{xxu} + e^{\sigma t}u^3\varphi_{yyu} = 0,$

$-\eta_{xx} + \eta_{yy} + 2\varphi_{yy} = 0,$

$\varphi_{xx} + \eta_{yy} - 2\varphi_{xu} = 0,$

$-u^3\eta_x - 2e^{\sigma t}\varphi_{ytu} = 0,$

$\xi_y + \eta_x = 0,$ (12)

III. Reduction of one dimensional Abelian Sub-algebra for $\mathbf{u_t + u^3u_x + u - e^{\sigma t}(u_{xxt} + u_{yyt}) = 0}$

After some more manipulations one finds that $\eta and \xi$ becomes

$$\xi = K_1 \tag{13}$$

$$\eta = K_3 \tag{14}$$

The remaining equations can then be used to determine τ and φ as

$$\tau = K_2 \tag{15}$$

$$\varphi = 0 \tag{16}$$

At this stage we construct the symmetry generators corresponding to each of the constants involved. These are a total of eight generators given by

$$V_1 = \partial_x$$

$$V_2 = \partial_t$$

$$V_3 = \partial_y \tag{17}$$

The one-parameter groups $g_i(\in)$ generalized by the V_i, where i=1, 2, 3. Are

$g_1(\in) : (x, y, t; u) \rightarrow (x + \in, y, t, u)$,

$g_2(\in) : (x, y, t; u) \rightarrow (x, y, t + \in, u)$,

$g_3(\in) : (x, y, t; u) \rightarrow (x, y + \in, t, u)$,

Where $\exp(\in V_i)(x, y, t; u) = (\bar{x}, \bar{y}, \bar{t}; \bar{u})$ and

(i) g_2 is time translation ,

(ii) g_1 and g_3 are the space-invariant of the equation

It is easy to check that the symmetry generators found in (10) form a closed Lie algebra whose communication relations are given in Table 1

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	0
V_2	0	0	0	0
V_3	0	0	0	0
V_4	0	0	0	0

Commutation relations satisfied by generators

The commutation relations of the Lie algebra, determined by V_1, V_2, V_3 and V_4 are shown in the above table. For this four dimensional Lie algebra the commutator table for V_i is a (4$\oplus$ 4) table whose $(i, j)^{th}$ entry expresses the Lie Bracket $[V_i, V_j]$ given by the above Lie algebra L . The table is skew-symmetric and the diagonal elements all vanish. The coefficient $c_{i,j,k}$ is the coefficient of Vi of the $(i, j)^{th}$ entry of the commutator table. The Lie algebra L is solvable. In the next section, we derive the reduction of (6) to PDEs with two independent variables and ODEs. These are four one-dimensional Lie subalgebras

$$L_{S,1} = \{V_1\} \qquad L_{S,2} = \{V_2\} \qquad L_{S,3} = \{V_3\} \qquad L_{S,4} = \{V_4\}$$

and corresponding to each one-dimensional subalgebras we may reduce (6)to a PDE with two independent variables. Further reductions to ODEs are associated with two-dimensional subalgebras. It is evident from the commutator table that there is no two-dimensional solvable non-abelian subalgebras

IV. Reductions for One-Dimensional Subalgebra $\mathbf{u_t + u^3 u_x + u - e^{\sigma t}(u_{xxt} + u_{yyt}) = 0}$

Case 1 : $V_1 = \partial_x$

The characteristic equation associated with this generator is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}$$

We integrate the characteristic equation to get three similarity variables,

$$y = r, \quad t = s, \quad u = W(r, s) \tag{18}$$

Using these similarity variables in Equation (6) can be recast in the form,

$$W_s + W - e^{\sigma s} W_{rrs} = 0 \tag{19}$$

Case 2 : $V_2 = \partial_t$.

The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}$$

Following the standard procedure we integrate the characteristic equation to get three similarity variables

$$x = r, \quad y = s, \quad u = W(r, s) \tag{20}$$

Using these similarity variables in Equation (6) can be recast in the form,

$$W(W^2 W_r + 1) = 0 \tag{21}$$

Case 3 : $V_3 = \partial_y$.

The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0}$$

Following the standard procedure we integrate the characteristic equation to get three similarity variables

$$x = r, \quad t = s, \quad u = W(r, s) \tag{22}$$

Using these similarity variables in Equation (6) can be recast in the form,

$$W_s + W^3 W_r + W - e^{\sigma t} W_{rrs} = 0 \tag{23}$$

V. Reduction of two dimensional Abelian Subalgebra for $\mathbf{u_t + u^3 u_x + u - e^{\sigma t}(u_{xxt} + u_{yyt}) = 0}$

Case 1: Reduction under V_1 and V_2. From Table 1 we find that the given generators commute $[V_1, V_2] = 0$. Thus either of V_1 or V_2 can be used to start the reduction with. For our purpose we begin reduction with V_1. Therefore we get Equation (18) and Equation (19). At this stage, we express c in terms of the similarity variables defined in (18). The transformed V_2 is

$$\widetilde{V_2} = \partial s$$

The characteristic equation for $\widetilde{V_2}$ is,

$$\frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}$$

Integrating this equation as before leads to new variables

$$r = \alpha, \quad W = \beta(\alpha)$$

Which reduce Equation (19) to

$$\beta(\alpha) = 0 \tag{24}$$

Case 2: Reduction under V_1 and V_3. From Table 1 we find that the given generators commute $[V_1, V_3] = 0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our convenience we begin reduction with V_1. Therefore we get Equation (22) and Equation (23). At this stage, we express V_3 in terms of the similarity variables defined in Equation (22).

The transformed V_3 is

$$\widetilde{V_3} = \partial r$$

The characteristic equation for $\widetilde{V_3}$ is,

$$\frac{dr}{1} = \frac{ds}{0} = \frac{dw}{0}$$

Integrating this equation as before leads to new variables

$$s = \alpha, \quad W = \beta(\alpha)$$

Which reduce Equation (19) to

$$\beta' + \beta = 0 \tag{26}$$

Case 3: Reduction under V_3 and V_2. From Table 1 we find that the given generators commute $[V_3, V_2] = 0$. Thus either of V_2 or V_3 can be used to start the reduction with. For our convenience we begin reduction with V_3. At this stage, we express V_2 in terms of the similarity variables defined in Equation (20).

The transformed V_2 is,

$$\widetilde{V_2} = \partial s$$

The characteristic equation for $\widetilde{V_2}$ is,

$$\frac{dr}{0} = \frac{ds}{1} = \frac{dw}{0}$$

Integrating this equation as before leads to new variables

$$r = \alpha, \quad W = \beta(\alpha)$$

Which reduce Equation (23) to

$$\beta(\beta^2\beta' + 1) = 0 \tag{27}$$

Reductions in remaining cases using generators forming sub-algebra are given in the form of Table 2 in Appendix A.

APPENDIX A

Table 2

Algebra	**Reduction**
$[V_1, V_2]$	$\beta = 0$
$[V_1, V_3]$	$\beta' + \beta = 0$
$[V_2, V_1]$	$\beta = 0$
$[V_2, V_3]$	$\beta(\beta^2\beta' + 1) = 0$
$[V_3, V_1]$	$\beta' + \beta = 0$
$[V_3, V_2]$	$\beta(\beta^2\beta' + 1) = 0$

VI. CONCLUSION

In this paper, A (2+1)-dimensional modified Equal width wave equation with damping term,

$$u^3 u_x + u - e^{\sigma t}(u_{xxt} + u_{yyt}) = 0$$

Where $e^{\sigma t} \in \mathbb{R}$ is subjected to Lie's classical method. Equation (6) admits a three-dimensional symmetry group. (iii) It is established that the symmetry generators form a closed Lie algebra. Classification of symmetry algebra of (6) into one- and two-dimensional subalgebras is carried out. Systematic reduction to (1+1)-dimensional PDE and then to first order ODEs are performed using one-dimensional and two-dimensional solvable Abelian subalgebras

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