Convergence in a Metric Space

Pawan Kumar
Registration No. 01409212020
Research Scholar: Mathematics
Dravidian University, Kuppam (A.P.) Pin code – 517425(India)

Definition - Let X be a metric space with metric d and let $\{x_n\} = \{x_1, x_2, \ldots, x_n, \ldots\}$ be a sequence of points in X. We say that $\{x_n\}$ is convergent if $\exists$ a point $x \in X$ such that either

(i) for each $\varepsilon > 0$, $\exists$ a positive integer n_0 such that $d(x_n, x) < \varepsilon$, $\forall\ x \geq n_0$

OR

(ii) for each open sphere $S_\varepsilon(x)$ centred on x, $\exists$ a positive integer n_0 such that

$$x_n \in S_\varepsilon(x), \quad \forall\ n \geq n_0$$

We usually symbolize this by writing $x_n \to x$

The point x is called the limit of the sequence $\{x_n\}$ and we sometimes write

$$\lim_{n\to\infty} x_n = x$$

Theorem $\rightarrow$ Limit of a convergent sequence is always unique.

Proof$\rightarrow$ Let $\{x_n\}$ be a sequence in a metric space X converging to $x \in X$

i. e. $x_n \to x \in X$

Let, if possible, $x_n \to y \in X$ $(x \neq y)$

Consider, $d(x, y) \leq d(x, x_n) + d(x_n, y)$ (By triangle inequality)

$= d(x_n, x) + d(x_n, y)$ (By Symmetry)

$\to 0$ as $n \to \infty$

$$\left[\begin{array}{l} \because x_n \to x \Rightarrow d(x_n, x) \to 0 \quad as\ n \to \infty \\ \&\ x_n \to y \Rightarrow d(x_n, y) \to 0 \quad as\ n \to \infty \end{array}\right]$$

i.e. $d(x, y) \to 0$ as $n \to \infty$

$\Rightarrow x = y$

Therefore, the limit of a sequence is always unique.

Definition → Cauchy Sequence

A sequence { x_n } in a metric space X with metric d is said to be a Cauchy sequence if for any $\varepsilon > 0$, $\exists$ a positive integer n_0 such that

$$d(x_n, x_m) < \varepsilon \qquad \forall\, n, m \geq n_0$$

Theorem → Every convergent sequence is Cauchy.

Proof→ Let $\{x_n\}$ be a convergent sequence l_n a metric space X with metric d.

Suppose $x_n \to x \in X$

So, for a given $\varepsilon > 0$, $\exists$ a positive integer n_0 such that

$$d(x_n, x) < \frac{\varepsilon}{2} \qquad \forall\, n \geq n_o\,.$$

For $m, n \geq n_o$

$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m)$ (By triangle inequality)

$= d(x_n, x) + d(x_m, x)$ (By Symmetry)

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow \quad d(x_n, x_m) < \varepsilon \qquad \forall\, n, m \geq n_0$

$\therefore$ The sequence $\{x_n\}$ is a Cauchy sequence.

Remark → Converse of the above theorem is not true.

i.e. A Cauchy sequence need not be convergent.

Example → Let X = (0, 1) be the subspace of the real line. Let $\{x_n\}$ be a sequence in X defined by $x_n = \frac{1}{n}$

Proof→ First, we show that $\{x_n\}$ is a Cauchy sequence.

Now $d(x_n, x_m) = |x_n - x_m| = \left|\frac{1}{n} - \frac{1}{m}\right| \to 0$ as $n, m \to \infty$

$\Rightarrow$ $\{x_n\}$ is a Cauchy sequence in X.

Next, we claim that $\{x_n\}$ is not convergent in X.

Now $x_n = \frac{1}{n}$

Clearly, $\lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{n} \to 0$

But $0 \notin X = (0,1)$

Therefore, the given sequence is not convergent in X, though it is Cauchy in X.

Observations

1. Let $\{x_n\}$ be a sequence such that $x_n=1$ $\forall$ n

Solution → Given sequence is of the type

$$\{1, 1, 1, \ldots\ldots\ldots, 1, 1, \ldots\ldots\ldots\}$$

Now $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} 1 = 1$

∴ Limit of the sequence = 1 ……….(1)

Now, limit point of the set of points of the sequence $\{x_n\} = \{1,1,1,\ldots\ldots\}$

$= \{1\} = \phi$ …………..(2)

From equation (1) & (2), we get

Limit ≠ Limit point

2. Let $\{x_n\}$ be a sequence such that $x_n = \frac{1}{n}$ $\forall$ **n**

Solution→ Let the sequence $\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\}$.

Now, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$

∴ Limit of the sequence = 0 ……….(1)

Limit point of the set of the points of the sequence $\{x_n\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\ldots\}$

= 0 ……… (2)

From equation (1) & (2), we get

Limit = Limit Point.

Theorem → If a convergent sequence in a metric space has infinitely many distinct points, them its limit is a limit point of the set of the points of the sequence.

Proof → Let X be a metric space and let $\{x_n\}$ be a convergent sequence in X with limit $x \in X$

We assume that x is not a limit point of the set of the points of the sequence.

This imply that there exists an open sphere $S_r(x)$ centred on x, which contains no point of the sequence different from x.

However, since x is the limit of the sequence, all x_n 's form some place on must lie in $S_r(x)$. Hence, must concide with x. From this, we see that there are only finitely many distinct points in the sequence. But points are infinitely. (given)

Therefore, our assumption is wrong.

Therefore, x is a limit point of the set of the points of the sequence.

Hence, The proof.

Definition → Complete metric space.

A metric space is said to be complete if every Cauchy sequence in it is convergent.

For example, IR, ₵ are complete metric space.

Example → Let C [a,b] be the set of all the real valued continuous functions defined on [a,b]. i.e. $C[a,b] = [f \mid f : (a,b) \xrightarrow{\text{continuous}} IR]$

We claim that C [a,b] is a complete metric space.

Proof → We define in C [a,b]

$$d(f,g) = \max_{x \in [a,b]} |f(x) - g(x)|, \quad f,g \in C[a,b]$$

(i) Clearly, $d(f, g) \geq 0$ [$\because |f(x)-g(x)| \geq 0, \forall x \in [a,b]$].

Also, $d(f, g) = \max_{x\in[a,b]} |f(x) - g(x)| = 0$

iff $|f(x) - g(x)| = 0 \quad \forall x \in [a,b]$

iff $f(x) - g(x) = 0 \quad \forall x \in [a,b]$

iff $f(x) = g(x) \quad \forall x \in [a,b]$

iff $f = g$

Therefore, $d(f,g) = 0$ iff $f = g$

(ii) $d(f, g) = \max_{x\in[a,b]} |f(x) - g(x)| = \max_{x\in[a,b]} |(-1)g(x) - f(x)|$

$= \max_{x\in[a,b]} |g(x) - f(x)| = d(g,f)$

(iii) $d(f, g) = \max_{x\in[a,b]} |f(x) - g(x)|$

$= \max_{x\in[a,b]} |\{f(x) - h(x)\} + \{h(x) - g(x)\}|$ where $h \in C[a,b]$

$\leq \max_{x\in[a,b]} |\{f(x) - h(x)\}| + \max_{x\in[a,b]} |\{h(x) - g(x)\}|$

$= d(f,h) + d(h,g)$

Thus, $d(f,g) \leq d(f,h) + d(h,g)$ where $f,g,h \in C[a,b]$

⇒ d is a metric on C [a,b] and therefore C [a,b] is a metric space.

Next. We prove that C [a,b] is a complete metric space.

Let $\{f_n\}$be a Cauchy sequence in C [a,b].

⇒ for given $\varepsilon > 0$, there exists a positive integer N such that

$d(f_n, f_m) < \varepsilon \qquad \forall\ n, m \geq N.$

$\Rightarrow \max. |f_n(x) - f_m(x)| < \varepsilon \qquad \forall\ n, m \geq N\ \&\ \forall\ x \in [a,b].$

$\Rightarrow \{f_n(x)\}$is a uniformly Cauchy sequence.

$\Rightarrow \{f_n(x)\}$is a uniformly convergent.

Let$\{f_n(x)\}$be uniformly convergent to f(x).

i.e $\quad f_n(x) \to f(x)$

But we know that limit of a uniformly convergent sequence of continuous functions is again continuous.

$\therefore\ f \in C\,[a,b]$

Now $d(f_n, f) = \max\limits_{x\in[a,b]} |f_n(x) - f(x)|$

$\to 0 \quad$ as $n\to\infty$

$\Rightarrow f_n \to f \in C\,[a,b]$

Therefore, every Cauchy sequence in C [a,b] is convergent

Hence, C[a,b] is a complete metric space.

Hence, the prof.

Theorem → Prove that IR^n is a complete metric space.

Proof → $IR^n = IR \times IR \times$ ------$\times IR$ (n. times)

$= \{(x_1, x_2, -----, x_n) \mid x_i \in IR,\ 1 \leq i \leq n\}$

To prove that IR^n is a complete metric space, we first show that IR^n is a metric space. We define

$d : IR^n \times IR^n \to IR$ by

$d(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} \quad ; \quad x, y \in IR^n$

(i) Clearly, $d(x, y) \geq 0 \qquad [\because |x_i - y_i| \geq 0,\ \forall\ i,\ 1 \leq i \leq n]$

Also, d(x,y)=0 iff $\left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}} = 0$

iff $|x_i - y_i|^2 = 0 \qquad \forall\ i,\ 1 \leq i \leq n$

iff $x_i - y_i = 0 \qquad \forall\ i,\ 1 \leq i \leq n$

iff $x_i = y_i \qquad \forall\ i,\ 1 \leq i \leq n$

iff $x = y \qquad$ [As $x = (x_1, x_2, -----, x_n) = (y_1, y_2, ----, y_n) = y$]

(ii) $d(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{\frac{1}{2}}$

$= \left(\sum_{i=1}^{n}(-1)^2|y_i - x_i\ |^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{n}|y_i - x_i|^2\right)^{\frac{1}{2}}$

= d (y,x)

(iii) d (x, y) = $\left(\sum_{i=1}^{n}|x_i - y_i\ |^2\right)^{\frac{1}{2}}$

$= \left(\sum_{i=1}^{n}|(x_i - z_i) + (z_i - y_i)|^2\right)^{\frac{1}{2}}$, $z_i \in$ IR

Using Misokauski's Inequality, we have

d (x, y) ≤ $\left(\sum_{i=1}^{n}|x_i - z_i\ |^2\right)^{\frac{1}{2}} + \left(\sum_{i=1}^{n}|z_i - y_i\ |^2\right)^{\frac{1}{2}}$

⇒ d (x, y) ≤ d (x,z) + d(z, y)

Therefore, d is a metric on IR^n.

Hence, IR^n is a metric space.

Now, we claim that IR^n is complete.

Let $\{P_m\}$ be Cauchy sequence in IR^n.

Now, P_m = $\{P_1, P_2,$ -------, P_m, -----}

= $\{(x_1^1, x_2^1$----,$x_n^1), (x_1^2, x_2^2$----,$x_n^2)$,--------, $(x_1^m, x_2^m$----,$x_n^m)$, ------}

R.T.P. → $\{P_m\}$ is convergent in order to prove that IR^n is complete

[Now, we know that a sequence $\{(x_1^m, x_2^m$----,$x_n^m)$, }of points of IR^n is convergent to L., where L= $(x_1, x_2$, -----, $x_n)$, iff every coordinate sequence $\{x_i^m\}$is convergent to x_i, $1 \leq i \leq n$]

------------(A)

Since $\{P_m\}$ is a Cauchy sequence in IR^n

So, $\{x_i^m\}$is a Cauchy sequence in IR for i=1, 2, ------, n.

⇒ $\{x_i^m\}$ is convergent for i = 1, 2, ----, n. [∵ IR is complete]

Let $\lim_{m\to\infty} x_i^m = x_i$

⇒ $\{x_1^m, x_2^m$----,$x_n^m\}$ converges to $(x_1, x_2$, -----, $x_n)$ using (A)

Therefore, (P_m) converges in IR^n

Hence, IR^n is a complete metric space.

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