

# A STUDY OF GENERALIZED FRACTIONAL INTEGRA OPERATORS INVOLVING THE ALEPH FUNCTION ( $\aleph$ ) AND GENERALIZED WRIGHT'S HYPERGEOMETRIC FUNCTION

$$\left( {}_P\bar{\Psi}_Q \right)$$

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**Abstract:** The aim of the present paper is to introduce and study new generalized fractional integral operators involving the product of  $\aleph$ -function and generalized Wright's hypergeometric function. On account of the most general nature of the above functions, a large number of fractional integral operators introduced by several authors lying scattered in the literature follow as special cases of our findings. Thus our operators of study generalize the fractional integral operators given by Gupta et al. [2], Saxena and Kumbhat [17], Saigo [16], Kober [11], Riemann-Liouville [11]. Next, we obtain the images of  $\aleph$ -function in our operators of study. Finally, we obtain the images of simpler fractional integral operators involving several special functions notably the H-function, generalized hypergeometric function  $\left( {}_P\bar{F}_Q \right)$ , generalized Wright Bessel's function  $\left( \bar{J}_{\lambda}^{\nu, \mu} \right)$ , generalized Mittag-Leffler function  $\left( E_{\alpha, \beta}^{\rho} \right)$ , extended Hurwitz-Lerch Zeta function  $\left( \phi_{\eta, \delta, \zeta}^{\omega, \kappa, \gamma} \right)$ .

**2010 Mathematics Subject Classification:** fractional integral operators,  $\aleph$ -function, H-function, Mittag-Leffler function, generalized Wright's hypergeometric function, Bessel function, extended Hurwitz-Lerch Zeta function  $\left( \phi_{\eta, \delta, \zeta}^{\omega, \kappa, \gamma} \right)$ .

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## 1. Fractional Integral Operators

In this section, we introduce and study following two new generalized fractional integral operators involving the product of  $\aleph$ -function and generalized Wright's hypergeometric function

$$\begin{aligned} I_x^{\mu, \lambda} [f(t)] &= \frac{x^{-\mu - \lambda - 1}}{\Gamma(\lambda)} \int_0^x t^{\mu} (x-t)^{\lambda} {}_P\bar{\Psi}_Q \left[ z_0 \left( \frac{t}{x} \right)^{\mu_0} \left( 1 - \frac{t}{x} \right)^{\lambda_0} \middle| \begin{array}{l} (a_j, \alpha_j; A_j)_{1,P} \\ (b_j, \beta_j; B_j)_{1,Q} \end{array} \right] \\ &\times \aleph_{p_i, q_i, \tau_i; r}^{m_1, n_1} \left[ z' \left( \frac{t}{x} \right)^{\mu'} \left( 1 - \frac{t}{x} \right)^{\lambda'} \middle| \begin{array}{l} (c_j, \gamma_j)_{1, n_1}, \dots, [\tau_i(c_{ji}, \gamma_{ji})]_{n_1+1, p_i} \\ (d_j, \delta_j)_{1, m_1}, \dots, [\tau_i(d_{ji}, \delta_{ji})]_{m_1+1, q_i} \end{array} \right] f(t) dt \quad (1) \end{aligned}$$

and

$$\begin{aligned} {}_x J_x^{\mu, \lambda} [f(t)] &= \frac{x^\mu}{\Gamma(\lambda)} \int_x^\infty t^{-\lambda-\mu-1} (t-x)^\lambda {}_P \bar{\Psi}_Q \left[ z_0 \left( \frac{x}{t} \right)^{\mu_0} \left( 1 - \frac{x}{t} \right)^{\lambda_0} \middle| \begin{matrix} (a_j, \alpha_j; A_j)_{1,P} \\ (b_j, \beta_j; B_j)_{1,Q} \end{matrix} \right] \\ &\times {}_s \aleph_{p_i, q_i, \tau_i; r}^{m_1, n_1} \left[ z' \left( \frac{x}{t} \right)^{\mu'} \left( 1 - \frac{x}{t} \right)^{\lambda'} \middle| \begin{matrix} (c_j, \gamma_j)_{1, n_1}, \dots, [\tau_i(c_j, \gamma_j)]_{n_1+1, p_i} \\ (d_j, \delta_j)_{1, m_1}, \dots, [\tau_i(d_j, \delta_j)]_{m_1+1, q_i} \end{matrix} \right] f(t) dt \quad (2) \end{aligned}$$

In equations (1) and (2),  $f(t) \in A$ , where  $A$  stands for class of functions for which

$$f(t) = \begin{cases} o\{|t|^\zeta\} & , \max.\{|t|\} \rightarrow 0 \\ o\{|t|^{w_1} e^{-w_2|t|}\} & , \min.\{|t|\} \rightarrow \infty \end{cases}$$

The fractional integral operator defined by (1) is valid under the following conditions:

$$\begin{aligned} \operatorname{Re}(\mu + \zeta + 1) + \mu' \min_{1 \leq j \leq m_1} \operatorname{Re} \left( \frac{d_j}{\delta_j} \right) > 0 ; \quad \operatorname{Re}(\lambda + 1) + \lambda' \min_{1 \leq j \leq m_1} \operatorname{Re} \left( \frac{d_j}{\delta_j} \right) > 0 \\ \tau_i > 0, \mu' > 0, \lambda' > 0 \end{aligned}$$

The fractional integral operator defined by (2) is valid under the following conditions:

$$\operatorname{Re}(\lambda + 1) + \lambda' \min_{1 \leq j \leq m_1} \operatorname{Re} \left( \frac{d_j}{\delta_j} \right) > 0 ; \quad \tau_i > 0, \mu' > 0, \lambda' > 0$$

$$\operatorname{Re}(w_2) > 0 \quad \text{or} \quad \operatorname{Re}(w_2) = 0 ; \quad \operatorname{Re}(\mu - w_1) + \mu' \min_{1 \leq j \leq m_1} \operatorname{Re} \left( \frac{d_j}{\delta_j} \right) > 0$$

The Aleph-function occurring in equations (1) and (2) will be defined and represented in the following manner [5, 4]:

$$\aleph[z] = {}_s \aleph_{p_i, q_i, \tau_i; r}^{m, n} \left[ z \middle| \begin{matrix} (a_j, \alpha_j)_{1, n}, \dots, [\tau_i(a_j, \alpha_j)]_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, \dots, [\tau_i(b_j, \beta_j)]_{m+1, q_i} \end{matrix} \right] = \frac{1}{2\pi i} \int_L \Omega_{p_i, q_i, \tau_i; r}^{m, n}(s) z^{-s} ds \quad (3)$$

for all  $z \neq 0$ , where  $i = \sqrt{(-1)}$  and

$$\Omega_{p_i, q_i, \tau_i; r}^{m, n}(s) = \frac{\prod_{j=1}^m \Gamma(b_j + \beta_j s) \prod_{j=1}^n \Gamma(1 - a_j - \alpha_j s)}{\sum_{i=1}^r \tau_i \prod_{j=m+1}^n \Gamma(1 - b_j - \beta_j s) \prod_{j=n+1}^p \Gamma(a_j + \alpha_j s)} \quad (4)$$

The integration path  $L = L_{i\gamma\infty}$ , ( $\gamma \in R$ ) extends from  $\gamma - i\infty$  to  $\gamma + i\infty$ , and is such that the poles of  $\Gamma(1 - a_j - \alpha_j s)$ ,  $j = \overline{1, n}$  (the symbol  $\overline{1, n}$  is used for  $1, 2, \dots, n$ ) do not coincide with the poles of  $\Gamma(b_j + \beta_j s)$ ,  $j = \overline{1, m}$ . The parameters  $p_i, q_i$  are non-negative integers satisfying  $0 \leq n \leq p_i$ ,  $0 \leq m \leq q_i$ ,  $\tau_i > 0$  for  $i = \overline{1, r}$ . The parameters  $\alpha_j, \beta_j, \alpha_j, \beta_j$  are positive numbers and

$a_j, b_j, a_{ji}, b_{ji}$  are complex. All poles of the integrand (4) are assumed to be simple, and the empty product is interpreted as unity. The existence conditions for the defining integral (3) are given below:

$$\Psi_l > 0, |\arg z| < \frac{\pi}{2} \Psi_l, l = \overline{1, r} \quad (5)$$

$$\Psi_l \geq 0, |\arg z| < \frac{\pi}{2} \Psi_l \text{ and } \operatorname{Re}(\zeta_l) + 1 < 0 \quad (6)$$

where  $\Psi_l = \sum_{j=1}^n \alpha_j + \sum_{j=1}^m \beta_j - \tau_l \left( \sum_{j=n+1}^{p_l} \alpha_{jl} + \sum_{j=m+1}^{q_l} \beta_{jl} \right)$  (7)

and  $\zeta_l = \sum_{j=1}^m b_j - \sum_{j=1}^n a_j + \tau_l \left( \sum_{j=m+1}^{q_l} b_{jl} - \sum_{j=n+1}^{p_l} a_{jl} \right) + \frac{1}{2} (p_l - q_l); l = \overline{1, r}$  (8)

Recently, a number of useful functions lying scattered in the literature have been found which are special cases of the  $\mathfrak{N}$ -function. Now, we record such functions for the easy access of researchers in this field.

(i) For  $\tau_1 = \tau_2 = \dots = \tau_r = 1$  in (3), we get the I-function due to V.P.Saxena [14], defined in the following manner:

$$\mathfrak{N}_{p_i, q_i, 1; r}^{m, n} \left[ z \left| \begin{matrix} (a_j, \alpha_j)_{1, n}, \dots, (a_{ji}, \alpha_{ji})_{n+1, p_i} \\ (b_j, \beta_j)_{1, m}, \dots, (b_{ji}, \beta_{ji})_{m+1, q_i} \end{matrix} \right. \right] = I_{p_i, q_i; r}^{m, n}[z] = \frac{1}{2\pi i} \int_L \Omega_{p_i, q_i, 1; r}^{m, n}(s) z^{-s} ds \quad (9)$$

where the kernel  $\Omega_{p_i, q_i, 1; r}^{m, n}(s)$  can be defined from equation (4).

(ii) If we set  $r = 1$  in (9), it reduces to the familiar H-function introduced by Fox [19]:

$$\mathfrak{N}_{p_i, q_i, 1; 1}^{m, n} \left[ z \left| \begin{matrix} (a_j, \alpha_j)_{1, p} \\ (b_j, \beta_j)_{1, q} \end{matrix} \right. \right] = H_{p, q}^{m, n}[z] = \frac{1}{2\pi i} \int_L \Omega_{p_i, q_i, 1; 1}^{m, n}(s) z^{-s} ds \quad (10)$$

where the kernel  $\Omega_{p_i, q_i, 1; 1}^{m, n}(s)$  can be obtained from equation (4).

(iii) If we set  $m = n = p = 1, q = m + 1, a_1 = b_1 = 0, \alpha_1 = \beta_1 = 1, b_2 = 1 - \mu_j, \beta_2 = 1/\rho_j, z = -z$  in (10),

it reduces to the multiindex (m-tuple) Mittag-Leffler function [9]:

$$\begin{aligned} \mathfrak{N}_{1, m+1, 1; 1}^{1, 1} \left[ -z \left| \begin{matrix} (0, 1) \\ (0, 1), \left( 1 - \mu_j, \frac{1}{\rho_j} \right) \end{matrix} \right. \right] &= H_{1, m+1}^{1, 1} \left[ -z \left| \begin{matrix} (0, 1) \\ (0, 1), \left( 1 - \mu_j, \frac{1}{\rho_j} \right) \end{matrix} \right. \right] = E_{\frac{1}{\rho_j}, \mu_j}(z) \\ &= \sum_{n=0}^{\infty} \frac{z^n}{\prod_{j=1}^m \Gamma\left(\mu_j + \frac{n}{\rho_j}\right)_{1, m}} \end{aligned} \quad (11)$$

where  $m \geq 1, \rho_j > 0; j = \overline{1, m}$ .

Further, if we set  $m = 1$  in (11), it reduces to the Mittag-Leffler function [18]:

$$\mathbb{H}_{1,2,1;1}^{1,1} \left[ -z \middle| \begin{matrix} (0,1) \\ (0,1), \left( 1-\mu, \frac{1}{\rho} \right) \end{matrix} \right] = H_{1,2}^{1,1} \left[ -z \middle| \begin{matrix} (0,1) \\ (0,1), \left( 1-\mu, \frac{1}{\rho} \right) \end{matrix} \right] = E_{\frac{1}{\rho}, \mu}^{\rho} (z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma \left( \frac{1}{\rho} n + \mu \right)} = {}_1\Psi_1 \left[ z \middle| \begin{matrix} (1,1) \\ \left( \mu, \frac{1}{\rho} \right) \end{matrix} \right] \quad (12)$$

**(iv)** If we set  $m = n = p = 1, q = 2, a_1 = 1 - \rho, \alpha_1 = \beta_1 = 1, b_1 = 0, b_2 = 1 - \beta, \beta_2 = \alpha, z = -z$

in (10), it reduces to the generalized Mittag-Leffler function [21]:

$$\mathbb{H}_{1,2,1;1}^{1,1} \left[ -z \middle| \begin{matrix} (1-\rho,1) \\ (0,1), (1-\beta, \alpha) \end{matrix} \right] = H_{1,2}^{1,1} \left[ -z \middle| \begin{matrix} (1-\rho,1) \\ (0,1), (1-\beta, \alpha) \end{matrix} \right] = E_{\alpha, \beta}^{\rho} (z) \quad (13)$$

**(v)** If we set  $m = q = 2, n = p = 1, a_1 = 1 - \frac{1}{\alpha-1} + \frac{\nu}{\rho}, \alpha_1 = \beta_2 = \frac{\beta}{\rho}, b_1 = 0, \beta_1 = 1, b_2 = \frac{\nu}{\rho}$ ,

$z = [a(\alpha-1)]^{\beta/\rho} z$  in (10), then it reduces to the generalized Kratzel function

$$\begin{aligned} \mathbb{H}_{1,2,1;1}^{2,1} \left[ [a(\alpha-1)]^{\beta/\rho} z \middle| \begin{matrix} \left( 1 - \frac{1}{\alpha-1} + \frac{\nu}{\rho}, \frac{\beta}{\rho} \right) \\ (0,1), \left( \frac{\nu}{\rho}, \frac{\beta}{\rho} \right) \end{matrix} \right] &= H_{1,2}^{2,1} \left[ [a(\alpha-1)]^{\beta/\rho} z \middle| \begin{matrix} \left( 1 - \frac{1}{\alpha-1} + \frac{\nu}{\rho}, \frac{\beta}{\rho} \right) \\ (0,1), \left( \frac{\nu}{\rho}, \frac{\beta}{\rho} \right) \end{matrix} \right] \\ &= \rho [a(\alpha-1)]^{\nu/\rho} \Gamma \left( \frac{1}{\alpha-1} \right) D_{\rho, \beta}^{\nu, \alpha} (z) \end{aligned} \quad (14)$$

where  $\rho > 0, a > 0, \beta > 0, \alpha > 1, |\arg z| < \left( \frac{2\beta}{\rho} + 1 \right) \frac{\pi}{2}$  and  $D_{\rho, \beta}^{\nu, \alpha}$  stands for generalized Krätzel function [3, p. 610, Eq. (42)].

The Wright's generalized hypergeometric function  $P\bar{\Psi}_Q$  occurring in equations (1) and (2) will be defined and represented in the following manner [15, p. 19, Eq. (2.6.11)]:

$$P\bar{\Psi}_Q \left[ z \middle| \begin{matrix} (a_j, \alpha_j; A_j)_{1,P} \\ (b_j, \beta_j; B_j)_{1,Q} \end{matrix} \right] = \overline{H}_{P, Q+1}^{1, P} \left[ -z \middle| \begin{matrix} (1-a_j, \alpha_j; A_j)_{1,P} \\ (0,1), (1-b_j, \beta_j; B_j)_{1,Q} \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^P \{ \Gamma(a_j + \alpha_j n) \}^{A_j} z^n}{\prod_{j=1}^Q \{ \Gamma(b_j + \beta_j n) \}^{B_j} n!} \quad (15)$$

The  $\overline{H}$ -function occurring in the above equation (15) was introduced by Inayat Hussain [13] and studied by Buschman and Srivastava [12].

Now, we present some special cases of the Wright's generalized hypergeometric function  $P\bar{\Psi}_Q$ .

**(i)** If we take  $\alpha_j = \beta_j = 1$  in (15), it reduces to generalized hypergeometric function [7,

p.271, Eq. (9)]

$$P\bar{\Psi}_Q \left[ z \middle| \begin{matrix} (a_j, 1; A_j)_{1,P} \\ (b_j, 1; B_j)_{1,Q} \end{matrix} \right] = \overline{H}_{P, Q+1}^{1, P} \left[ -z \middle| \begin{matrix} (1-a_j, 1; A_j)_{1,P} \\ (0,1), (1-b_j, 1; B_j)_{1,Q} \end{matrix} \right] = \frac{\prod_{j=1}^P \{ \Gamma(a_j) \}^{A_j}}{\prod_{j=1}^Q \{ \Gamma(b_j) \}^{B_j}} {}_P\bar{F}_Q \left( \begin{matrix} (a_j; A_j) \\ (b_j; B_j) \end{matrix} \middle| z \right) \quad (16)$$

**(ii)** If we take  $P = 0, Q = 1, b_1 = 1 + \lambda, \beta_1 = \nu, B_1 = \mu, z = -z$  in (15), it reduces to the generalized Wright's Bessel function [7, p.271, Eq. (8)]

$$0\bar{\Psi}_1 \left[ -z \middle| \begin{matrix} - \\ (1+\lambda, \nu; \mu) \end{matrix} \right] = \overline{H}_{0, 2}^{1, 0} \left[ z \middle| \begin{matrix} - \\ (0,1), (-\lambda, \nu; \mu) \end{matrix} \right] = \bar{J}_{\lambda}^{\nu, \mu} = \sum_{r=0}^{\infty} \frac{(-z)^r}{r! \{ \Gamma(1+\lambda+\nu r) \}^{\mu}} \quad (17)$$

(iii) If we put

$$P = 3, Q = 2, a_1 = \eta, \alpha_1 = \omega, a_2 = \delta, \alpha_2 = \kappa, a_3 = b, A_1 = A_2 = \alpha_3 = 1, A_3 = B_2 = s,$$

$b_1 = \zeta, \beta_1 = \gamma, b_2 = 1 + b, B_1 = \beta_2 = 1$  in (15), it reduces to the extended Hurwitz Lerch Zeta function [10, p. 491, Eq. (1.20)]

$$\begin{aligned} {}_3\bar{\Psi}_2 \left[ \begin{matrix} (\eta, \omega; 1), (\delta, \kappa; 1), (b, 1; s) \\ (\zeta, \gamma; 1), (1+b, 1; s) \end{matrix} \right] &= \overline{H}_{3,3}^{1,3} \left[ \begin{matrix} (1-\eta, \omega; 1), (1-\delta, \kappa; 1), (1-b, 1; s) \\ (0, 1), (1-\zeta, \gamma; 1), (-b, 1; s) \end{matrix} \right] \\ &= \frac{\Gamma(\eta)\Gamma(\delta)}{\Gamma(\zeta)} \phi_{\eta, \delta, \zeta}^{\omega, \kappa, \gamma}(z, s, b) = \sum_{n=0}^{\infty} \frac{\Gamma(\eta + \omega n)\Gamma(\delta + \kappa n)}{\Gamma(\zeta + \gamma n)n!} \frac{z^n}{(b+n)^s} \end{aligned} \quad (18)$$

The  $\aleph$ -function of two variables occurring in the present paper will be defined and represented in the following manner [1, p. 149, Eqs. (9)- (12)]:

$$\begin{aligned} \aleph[x, y] &= \aleph^{o, n : m_1, n_1; m_2, n_2}_{p, q; p_i, q_i, \tau_i; p'_i, q'_i, \tau'_i; r} \\ &\left[ \begin{matrix} x \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1, p} : (c_j, \gamma_j)_{1, n_1}, \dots, [\tau_i(c_{ji}, \gamma_{ji})]_{n_1+1, p_i} ; (e_j, E_j)_{1, n_2}, \dots, [\tau'_i(e_{ji}, E_{ji})]_{n_2+1, p'_i} \\ y \left| \begin{matrix} (b_j, \beta_j; B_j)_{1, q} : (d_j, \delta_j)_{1, m_1}, \dots, [\tau_i(d_{ji}, \delta_{ji})]_{m_1+1, q_i} ; (f_j, F_j)_{1, m_2}, \dots, [\tau'_i(f_{ji}, F_{ji})]_{m_2+1, q'_i} \end{matrix} \right. \end{matrix} \right. \end{matrix} \right] \\ &= \frac{1}{(2\pi i)^2} \int_L \int_L \phi(\xi, \eta) \theta_1(\xi) \theta_2(\eta) x^{-\xi} y^{-\eta} d\xi d\eta \end{aligned} \quad (19)$$

$$\text{where } \phi(\xi, \eta) = \frac{\prod_{j=1}^n \Gamma(1-a_j - \alpha_j \xi - A_j \eta)}{\prod_{j=n+1}^p \Gamma(a_j + \alpha_j \xi + A_j \eta) \prod_{j=1}^q \Gamma(1-b_j - \beta_j \xi - B_j \eta)} \quad (20)$$

$$\theta_1(\xi) = \Omega_{p_i, q_i, \tau_i; r}^{m_1, n_1}(\xi) = \frac{\prod_{j=1}^{m_1} \Gamma(d_j + \delta_j \xi) \prod_{j=1}^{n_1} \Gamma(1-c_j - \gamma_j \xi)}{\sum_{i=1}^r \tau_i \prod_{j=m_1+1}^{q_i} \Gamma(1-d_{ji} - \delta_{ji} \xi) \prod_{j=n_1+1}^{p_i} \Gamma(c_{ji} + \gamma_{ji} \xi)} \quad (21)$$

$$\theta_2(\eta) = \Omega_{p'_i, q'_i, \tau'_i; r}^{m_2, n_2}(\eta) = \frac{\prod_{j=1}^{m_2} \Gamma(f_j + F_j \eta) \prod_{j=1}^{n_2} \Gamma(1-e_j - E_j \eta)}{\sum_{i=1}^r \tau'_i \prod_{j=m_2+1}^{q'_i} \Gamma(1-f_{ji} - F_{ji} \eta) \prod_{j=n_2+1}^{p'_i} \Gamma(e_{ji} + E_{ji} \eta)} \quad (22)$$

## 2. Special Case of Our Main Fractional Integral Operators Defined by (1) and (2)

(i) If we reduce  $\aleph$ -function occurring in (1) and (2) to (m-tuple) multiindex Mittag-Leffler function [9], we get the following fractional integral operator

$$I_x^{\mu, \lambda} [f(t)] = \frac{x^{-\mu - \lambda - 1}}{\Gamma(\lambda)} \int_0^x t^\mu (x-t)^\lambda P \bar{\Psi} Q \left[ z_0 \left( \frac{t}{x} \right)^{\mu_0} \left( 1 - \frac{t}{x} \right)^{\lambda_0} \left| \begin{matrix} (a_j, \alpha_j; A_j)_{1, P} \\ (b_j, \beta_j; B_j)_{1, Q} \end{matrix} \right. \right]$$

$$\times {}_x^E \frac{1}{\alpha_j} {}_{\beta_j} \left( z' \left( \frac{t}{x} \right)^{\mu'} \left( 1 - \frac{t}{x} \right)^{\lambda'} \right) f(t) dt \quad (23)$$

$$J_x^{\mu, \lambda} [f(t)] = \frac{x^\mu}{\Gamma(\lambda)} \int_x^\infty t^{-\lambda - \mu - 1} (t-x)^\lambda P \bar{\Psi} Q \left[ z_0 \left( \frac{x}{t} \right)^{\mu_0} \left( 1 - \frac{x}{t} \right)^{\lambda_0} \middle| \begin{matrix} (a_j, \alpha_j; A_j)_{1,P} \\ (b_j, \beta_j; B_j)_{1,Q} \end{matrix} \right]$$

$$\times {}_x^E \frac{1}{\alpha_j} {}_{\beta_j} \left( z' \left( \frac{x}{t} \right)^{\mu'} \left( 1 - \frac{x}{t} \right)^{\lambda'} \right) f(t) dt \quad (24)$$

**(ii)** If we reduce Wright's generalized hypergeometric function to extended Hurwitz-Lerch Zeta function and  $\aleph$ -function reduces to generalized Krätsel function in equations (1) and (2), we get the fractional integral operators recently obtained by Gupta et al. [2, p. 345, Eq. (24)]:

**(iii)** If in (23) and (24), we reduce Wright's generalized hypergeometric function to Gauss hypergeometric function by taking  $\mu_0 = 0, \lambda_0 = 1$  and multiindex (m-tuple) Mittag-Leffler function to unity, we obtain in essence the known fractional integral operators obtained by Saxena and Kumbhat [17]:

$$I_x^{\mu, \lambda} [f(t)] = \frac{x^{-\mu - \lambda}}{\Gamma(\lambda)} \int_0^x t^\mu (x-t)^{\lambda-1} {}_2 F_1 \left[ \eta, \delta; \zeta; z_0 \left( 1 - \frac{t}{x} \right) \right] f(t) dt \quad (25)$$

$$J_x^{\mu, \lambda} [f(t)] = \frac{x^\mu}{\Gamma(\lambda)} \int_x^\infty t^{-\lambda - \mu} (t-x)^\lambda {}_2 F_1 \left[ \eta, \delta; \zeta; z_0 \left( 1 - \frac{x}{t} \right) \right] f(t) dt \quad (26)$$

**(iv)** Further, if we take  $\mu = 0$  and make some suitable adjustments in (25) and (26), we arrive at the fractional integral operators studied by Saigo [16]

$$I_x^{\eta, \delta, \sigma} [f(t)] = \frac{x^{-\eta - \delta}}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} {}_2 F_1 \left[ \eta + \delta, -\sigma; \eta; \left( 1 - \frac{t}{x} \right) \right] f(t) dt \quad (27)$$

$$J_x^{\eta, \delta, \sigma} [f(t)] = \frac{1}{\Gamma(\eta)} \int_x^\infty t^{-\eta - \delta} (t-x)^{\eta-1} {}_2 F_1 \left[ \eta + \delta, -\sigma; \eta; \left( 1 - \frac{x}{t} \right) \right] f(t) dt \quad (28)$$

Further, if we put  $\delta = 0$  in (27) and (28), we get known fractional integral operators obtained by Kober [11, p. 322, Eqs. (18.5) & (18.6)].

Again, If we put  $\delta = -\eta$  in (27) and (28), we get fractional integral operators obtained by Riemann and Liouville [11, p. 94, Eqs. (5.1) & (5.3)].

### 3. Images

In this section, we find the images of the  $\aleph$ -function in our fractional integral operators defined by (1) and (2), we have

$$(i) \quad I_x^{\mu, \lambda} \left[ {}_t^{\sigma} \aleph_{p_i', q_i', \tau_i'}^{m_2, n_2} \right] = {}_r \left[ z(t)^{\mu_1(x-t)} {}_{\lambda_1} \left| \begin{matrix} (e_j, E_j)_{1, n_2}, \dots, [\tau_i'(e_j, E_j)]_{n_2+1, p_i'} \\ (f_j, F_j)_{1, m_2}, \dots, [\tau_i'(f_j, F_j)]_{m_2+1, q_i'} \end{matrix} \right. \right]$$

$$= \frac{x^\sigma}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^P \left\{ \Gamma(a_j + \alpha_j n) \right\}^{A_j} (z_0)^n}{\prod_{j=1}^Q \left\{ \Gamma(b_j + \beta_j n) \right\}^{B_j} n!}$$

$$\aleph_{2,1: p_i, q_i, \tau_i; p'_i, q'_i, \tau'_i; r}^{0,2: m_1, n_1; m_2, n_2} \left[ z' \begin{matrix} A^* : A_1^*; A_2^* \\ B^* : B_1^*; B_2^* \end{matrix} \middle| z(x)^{\mu_1 + \lambda_1} \right] \quad (29)$$

where  $A^* = (-\mu - \sigma - \mu_0 n; \mu', \mu_1), (-\lambda - \lambda_0 n; \lambda', \lambda_1); B^* = (-1 - \mu - \sigma - \lambda - (\lambda_0 + \mu_0) n; (\lambda' + \mu'), (\lambda_1 + \mu_1))$

$$A_1^* = (c_j, \gamma_j)_{1, n_2}, \dots, [\tau'_i (c_{ji}, \gamma_{ji})]_{n_2+1, p'_i}; B_1^* = (d_j, \delta_j)_{1, m_2}, \dots, [\tau'_i (d_{ji}, \delta_{ji})]_{m_2+1, q'_i}, \quad (30)$$

$$A_2^* = (e_j, E_j)_{1, n_2}, \dots, [\tau'_i (e_{ji}, E_{ji})]_{n_2+1, p'_i}; B_2^* = (f_j, F_j)_{1, m_2}, \dots, [\tau'_i (f_{ji}, F_{ji})]_{m_2+1, q'_i}, \quad (31)$$

provided that

$$\operatorname{Re}(\mu + \sigma) + \mu' \min_{1 \leq j \leq m_1} \operatorname{Re}(d_j / \delta_j) + \mu_1 \min_{1 \leq j \leq m_2} \operatorname{Re}(f_j / F_j) + 1 > 0 ;$$

$$\operatorname{Re}(\lambda) + \lambda' \min_{1 \leq j \leq m_1} \operatorname{Re}(d_j / \delta_j) + \lambda_1 \min_{1 \leq j \leq m_2} \operatorname{Re}(f_j / F_j) + 1 > 0 ; \tau_i > 0, \tau'_i > 0, \mu' > 0, \lambda' > 0$$

$$(ii) \quad J_x^{\mu, \lambda} \left[ t^\sigma \aleph_{p'_i, q'_i, \tau'_i; r}^{m_2, n_2} \left[ z(t)^{-\mu_1} \left( 1 - \frac{x}{t} \right)^{\lambda_1} \begin{matrix} (e_j, E_j)_{1, n_2}, \dots, [\tau'_i (e_{ji}, E_{ji})]_{n_2+1, p'_i} \\ (f_j, F_j)_{1, m_2}, \dots, [\tau'_i (f_{ji}, F_{ji})]_{m_2+1, q'_i} \end{matrix} \right] \right]$$

$$= \frac{x^\sigma}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^P \left\{ \Gamma(a_j + \alpha_j n) \right\}^{A_j} (z_0)^n}{\prod_{j=1}^Q \left\{ \Gamma(b_j + \beta_j n) \right\}^{B_j} n!}$$

$$\aleph_{2,1: p_i, q_i, \tau_i; p'_i, q'_i, \tau'_i; r}^{0,2: m_1, n_1; m_2, n_2} \left[ z' \begin{matrix} A^{**} : A_1^*; A_2^* \\ B^{**} : B_1^*; B_2^* \end{matrix} \middle| z(x)^{-\mu_1} \right] \quad (32)$$

where  $A_1^*, B_1^*, A_2^*$  and  $B_2^*$  are given in (30) and (31).

$$A^{**} = (1 + \sigma - \mu - \mu_0 n; \mu', \mu_1), (-\lambda - \lambda_0 n; \lambda', \lambda_1); B^{**} = (\sigma - \mu - \lambda - (\lambda_0 + \mu_0) n; (\lambda' + \mu'), (\lambda_1 + \mu_1))$$

provided that

$$\operatorname{Re}(\lambda) + \lambda' \min_{1 \leq j \leq m_1} \operatorname{Re}(d_j / \delta_j) + \lambda_1 \min_{1 \leq j \leq m_2} \operatorname{Re}(f_j / F_j) + 1 > 0 ; \tau_i > 0, \tau'_i > 0, \mu' > 0, \lambda' > 0$$

$$\operatorname{Re}(\mu - \sigma) + \mu' \min_{1 \leq j \leq m_1} \operatorname{Re}(d_j / \delta_j) + \mu_1 \min_{1 \leq j \leq m_2} \operatorname{Re}(f_j / F_j) > 0 ;$$

**Proof:** To prove (29), first of all we express the I-operator involved in its left hand side in the integral form with the help of (1). Then, we express generalized Wright's hypergeometric function in terms of series by using (15) and both Aleph- functions in contour forms by using (3). Next, we interchange the order of summation, contour integral with t- integral. Thus the left hand side of (31) assumes the following form (say  $\Delta$ )

$$\Delta = \frac{x^{-\mu-\lambda-1}}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^P \left\{ \Gamma(a_j + \alpha_j n) \right\}^{A_j} (z_0)^n}{\prod_{j=1}^Q \left\{ \Gamma(b_j + \beta_j n) \right\}^{B_j} n!} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \frac{\prod_{j=1}^{m_1} \Gamma(d_j + \delta_j \xi)}{\prod_{i=1}^r \tau_i \sum_{j=m_1+1}^{q_i} \Gamma(1-d_{ji} - \delta_{ji} \xi)} \frac{\prod_{j=1}^{n_1} \Gamma(1-c_j - \gamma_j \xi)}{\prod_{j=n_1+1}^{p_i} \Gamma(c_{ji} + \gamma_{ji} \xi)} \\ \times \frac{\prod_{j=1}^{m_2} \Gamma(f_j + F_j \eta)}{\prod_{i=1}^r \tau_i' \sum_{j=m_2+1}^{q_i'} \Gamma(1-f_{ji} - F_{ji} \eta)} \frac{\prod_{j=1}^{n_2} \Gamma(1-e_j - E_j \eta)}{\prod_{j=n_2+1}^{p_i'} \Gamma(e_{ji} + E_{ji} \eta)} (z')^{-\xi} (z)^{-\eta} x^{-(\mu_0+\lambda_0)n+(\mu'+\lambda')\xi} \\ \times \left\{ \int_0^x t^{\mu+\sigma+\mu_0 n - \mu' \xi - \mu_1 \eta} (x-t)^{\lambda+\lambda_0 n - \lambda' \xi - \lambda_1 \eta} dt \right\} d\xi d\eta$$

Now, we evaluate the t-integral occurring in above equation, we get

$$\Delta = \frac{x^\sigma}{\Gamma(\lambda)} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^P \left\{ \Gamma(a_j + \alpha_j n) \right\}^{A_j} (z_0)^n}{\prod_{j=1}^Q \left\{ \Gamma(b_j + \beta_j n) \right\}^{B_j} n!} \frac{1}{(2\pi i)^2} \int_{L_1} \int_{L_2} \frac{\Gamma(1+\mu+\sigma+\mu_0 n - \mu' \xi - \mu_1 \eta)}{\Gamma(1+\lambda+\lambda_0 n - \lambda' \xi - \lambda_1 \eta)} \frac{\prod_{j=1}^{m_1} \Gamma(d_j + \delta_j \xi)}{\prod_{j=1}^{n_1} \Gamma(1-c_j - \gamma_j \xi)} \\ \times \frac{\Gamma(2+\lambda+\mu+\sigma+(\lambda_0+\mu_0)n - (\lambda'+\mu')\xi - (\lambda_1+\mu_1)\eta)}{\Gamma\left(\sum_{i=1}^r \tau_i \sum_{j=m_1+1}^{q_i} \Gamma(1-d_{ji} - \delta_{ji} \xi)\right)} \frac{\prod_{i=1}^r \tau_i \sum_{j=m_1+1}^{q_i} \Gamma(1-d_{ji} - \delta_{ji} \xi)}{\prod_{j=n_1+1}^{p_i} \Gamma(c_{ji} + \gamma_{ji} \xi)} \\ \times \frac{\prod_{j=1}^{m_2} \Gamma(f_j + F_j \eta)}{\prod_{i=1}^r \tau_i' \sum_{j=m_2+1}^{q_i'} \Gamma(1-f_{ji} - F_{ji} \eta)} \frac{\prod_{j=1}^{n_2} \Gamma(1-e_j - E_j \eta)}{\prod_{j=n_2+1}^{p_i'} \Gamma(e_{ji} + E_{ji} \eta)} (z')^{-\xi} (z(x)^{\mu_1+\lambda_1})^{-\eta} d\xi d\eta \quad (33)$$

Finally, on interpreting the right hand side (33) in terms of  $\aleph$ -function of two variables defined by (19), we get the required result after a little simplification.

The proof of (32) can be obtained by proceeding on similar lines given to those given above.

#### 4. Applications of Image (i)

- (i) If in (29), we reduce  $P\bar{\psi}_Q$  to  $P\bar{F}_Q$  defined by (16) and both  $\aleph$ -functions to H-functions defined by (10), we get the following integral

$$\int_0^x t^{\mu+\sigma} (x-t)^\lambda P\bar{F}_Q \left[ z_0 \left( \frac{t}{x} \right)^{\mu_0} \left( 1 - \frac{t}{x} \right)^{\lambda_0} \begin{matrix} (a_j; A_j)_{1,P} \\ (b_j; B_j)_{1,Q} \end{matrix} \right] \\ H_{p,q}^{m_1, n_1} \left[ z' \left( \frac{t}{x} \right)^{\mu'} \left( 1 - \frac{t}{x} \right)^{\lambda'} \begin{matrix} (c_j, \gamma_j)_{1,p} \\ (d_j, \delta_j)_{1,q} \end{matrix} \right] \\ H_{p,q}^{m_2, n_2} \left[ z(t)^{\mu_1} (x-t)^{\lambda_1} \begin{matrix} (e_j, E_j)_{1,p'} \\ (f_j, F_j)_{1,q'} \end{matrix} \right] dt$$

$$= x^{\mu+\sigma+\lambda+1} \sum_{n=0}^{\infty} \frac{\prod_{j=1}^P \left\{ \Gamma(a_j + n) \right\}^{A_j} (z_0)^n}{\prod_{j=1}^Q \left\{ \Gamma(b_j + n) \right\}^{B_j} n!} H_{2,1: p,q; p',q'}^{0,2:m_1,n_1;m_2,n_2} \left[ z'(x)^{\mu_1+\lambda_1} \right] \\ \left. \begin{aligned} & (-\mu - \sigma - \mu_0 n; \mu', \mu_1), (-\lambda - \lambda_0 n; \lambda', \lambda_1) : (c_j, \gamma_j)_{1,p}; (e_j, E_j)_{1,p'} \\ & (-1 - \mu - \sigma - \lambda - (\mu_0 + \lambda_0) n; (\lambda' + \mu'), (\lambda_1 + \mu_1)) : (d_j, \delta_j)_{1,q}; (f_j, F_j)_{1,q'} \end{aligned} \right]$$

provided that

$$\operatorname{Re}(\mu + \sigma) + \mu' \min_{1 \leq j \leq m_1} \operatorname{Re}(d_j / \delta_j) + \mu_1 \min_{1 \leq j \leq m_2} \operatorname{Re}(f_j / F_j) + 1 > 0 ;$$

$$\operatorname{Re}(\lambda) + \lambda' \min_{1 \leq j \leq m_1} \operatorname{Re}(d_j / \delta_j) + \lambda_1 \min_{1 \leq j \leq m_2} \operatorname{Re}(f_j / F_j) + 1 > 0 ; \mu' > 0, \lambda' > 0$$

**(ii)** If in (29), we reduce  $P\bar{\psi}_Q$  to  $\bar{J}_{\lambda}^{\nu, \mu}$  defined by (17),  $\aleph$ -function  $\left( \aleph_{p_i, q_i, \tau_i; r}^{m_1, n_1} \right)$  to generalized Mittag-Leffler function defined by (13) and  $\aleph$ -function  $\left( \aleph_{p'_i, q'_i, \tau'_i; r}^{m_2, n_2} \right)$  to H-function defined by (10),

Mittag-Leffler function defined by (13) and  $\aleph$ -function  $\left( \aleph_{p'_i, q'_i, \tau'_i; r}^{m_2, n_2} \right)$  to H-function defined by (10), we get

$$\int_0^x t^{\mu+\sigma} (x-t)^{\lambda} \bar{J}_{\lambda}^{\nu, \mu} \left[ z_0 \left( \frac{t}{x} \right)^{\mu_0} \left( 1 - \frac{t}{x} \right)^{\lambda_0} \right] E_{\alpha, \beta}^{\rho} \left[ z' \left( \frac{t}{x} \right)^{\mu'} \left( 1 - \frac{t}{x} \right)^{\lambda'} \right] \\ H_{p, q}^{m_2, n_2} \left[ z(t)^{\mu_1} (x-t)^{\lambda_1} \left| \begin{array}{l} (e_j, E_j)_{1, p'} \\ (f_j, F_j)_{1, q'} \end{array} \right. \right] dt \\ = x^{\mu+\sigma+\lambda+1} \sum_{n=0}^{\infty} \frac{(z_0)^n}{\left\{ \Gamma(1+\lambda+\nu n) \right\}^{\mu} n!} H_{2,1:1,1;m_2,n_2}^{0,2:1,1;m_2,n_2} \left[ \begin{array}{l} -z' \\ z(x)^{\mu_1+\lambda_1} \end{array} \right] \\ \left. \begin{aligned} & (-\mu - \sigma - \mu_0 n; \mu', \mu_1), (-\lambda - \lambda_0 n; \lambda', \lambda_1) : (1-\rho, 1)_{1,p}; (e_j, E_j)_{1,p'} \\ & (-1 - \mu - \sigma - \lambda - (\mu_0 + \lambda_0) n; (\lambda' + \mu'), (\lambda_1 + \mu_1)) : (0, 1)_{1,q}; (f_j, F_j)_{1,q'} \end{aligned} \right]$$

provided that

$$\operatorname{Re}(\mu + \sigma) + \mu_1 \min_{1 \leq j \leq m_2} \operatorname{Re}(f_j / F_j) + 1 > 0 ; \quad \operatorname{Re}(\lambda) + \lambda_1 \min_{1 \leq j \leq m_2} \operatorname{Re}(f_j / F_j) + 1 > 0 ; \mu' > 0, \lambda' > 0 \quad \text{(iii)}$$

If in (29), we reduce  $P\bar{\psi}_Q$  to  $\phi_{\eta, \delta, \zeta}^{\omega, \kappa, \gamma}$  defined by (18),  $\aleph$ -function  $\left( \aleph_{p_i, q_i, \tau_i; r}^{m_1, n_1} \right)$  to generalized Krätszel

function defined by (14) and  $\aleph$ -function  $\left( \aleph_{p'_i, q'_i, \tau'_i; r}^{m_2, n_2} \right)$  to Mittag-Leffler function defined by (12), we

get

$$\int_0^x t^{\mu+\sigma} (x-t)^{\lambda} \phi_{\eta, \delta, \zeta}^{\omega, \kappa, \gamma} \left[ z_0 \left( \frac{t}{x} \right)^{\mu_0} \left( 1 - \frac{t}{x} \right)^{\lambda_0}, s, b \right] D_{\rho, \beta}^{\nu, \alpha} \left[ z' \left( \frac{t}{x} \right)^{\mu'} \left( 1 - \frac{t}{x} \right)^{\lambda'} \right]$$

$$\begin{aligned}
& E_{l,\varepsilon} \left[ z(t)^{\mu_1} (x-t)^{\lambda_1} \right] dt \\
&= x^{\mu+\sigma+\lambda+1} \frac{[a(\alpha-1)]^{-\nu/\rho}}{\rho \Gamma(1/\alpha-1)} \sum_{n=0}^{\infty} \frac{(\eta)_{\omega n} (\delta)_{\kappa n} (z_0)^n}{(\zeta)_n (b+n)^s n!} H_{2,1:1,2;1,2} \left[ \begin{matrix} [a(\alpha-1)]^{\beta/\rho} z \\ -z(x)^{\mu_1+\lambda_1} \end{matrix} \middle| \begin{matrix} (-\mu-\sigma-\mu_0 n; \mu', \mu_1), (-\lambda-\lambda_0 n; \lambda', \lambda_1) : \left(1 - \frac{1}{\alpha-1} + \frac{\nu}{\rho}, \frac{\beta}{\rho}\right); (0,1) \\ (-1-\mu-\sigma-\lambda-(\mu_0+\lambda_0)n; (\lambda'+\mu'), (\lambda_1+\mu_1)) : (0,1), \left(\frac{\nu}{\rho}, \frac{\beta}{\rho}\right); (0,1), (1-\varepsilon, l) \end{matrix} \right]
\end{aligned}$$

provided that

$$\alpha > 1 ; \operatorname{Re}(\mu+\sigma+1) > 0 ; \operatorname{Re}(\lambda+1) > 0 ;$$

$$\min. [\omega, \kappa, \gamma, a, \rho, \beta, \mu', \lambda'] \geq 0 \text{ (not all simultaneously zero) .}$$

### References

- [1] Saxena, R.K., Ram, J. and Kumar, D. (2013). Generalized Fractional Integration of the Product of Two  $\aleph$ -Functions Associated with the Appell Function  $F_3$ . *Romai J.*, **9**(1), pp. 147–158.
- [2] Gupta, K.C., Jain, R. and Kumawat, P. (2012): A Study of New Fractional Integral Operators Involving Extended Hurwitz-Lerch Zeta Function and Generalized Krätzel Function. *J. Raj. Acad. Phy. Sci.*, **11**(4), pp. 339-352.
- [3] Kumar, D. (2011): P-Transform. *Integral Trans. & Special Functions*, **22**(8), pp. 603-616.
- [4] Saxena, R.K. and Pogány, T.K. (2011). On Fractional Integration Formulae for Aleph Functions. *Appl. Math. Comput.*, **218**, pp. 985-990.
- [5] Saxena, R.K. and Pogány, T.K. (2010). Mathieu-Type Series for the  $\aleph$ -Function Occurring in Fokker-Planck Equation. *EJPAM*, **3**(6), pp. 980-988.
- [6] Gupta, K.C. and Soni, R.C. (2004): A Unified Laplace Transform Formula, Functions of Practical Importance and H-Function. *J. Raj. Acad. Phy. Sci.*, **1**(1), pp. 7-16.
- [7] Gupta, K.C., Jain, R. and Sharma, A. (2003): A Study of Unified Finite Integral Transforms with Applications. *J. Raj. Acad. Phy. Sci.*, **2**(4), pp. 269-282.
- [8] Mainardi, F., Luchko, Y. and Pagnini, G. (2001): The Fundamental Solution of the Space Time Fractional Diffusion Equation. *Fractional Calculus and Applied Analysis*, **4**(2), pp. 153-192.
- [9] Kiryakova, V.S. (2000): Multiple (multi index) Mittag-Leffler Functions and Relation to Generalized Fractional Calculus. *J. Compt. Appl. Math.*, **118**, pp. 241-259.
- [10] Srivastava, H.M., Saxena, R.K., Pogany, t.K. and Saxena, R.K. (1999): Integral and Computational Representation of the Extended Hurwitz –Lerch Zeta Function. *Integral Trans. & Special Functions*, **22** (7), pp. 487-506.
- [11] Samko, S. G., Kilbas, A. A. and Marichev, O. I., (1993): “Fractional Integrals and Derivatives, Theory and Applications.” *Gordon and Breach Sci. Publ., New York*.

- [12] **Buschman, R.G. and Srivastava, H.M. (1990):** The  $\bar{H}$  - Function Associated with a Certain Class of Feynman Integrals, *J. Phys. A: Math. Gen.* **23**, pp. 4707-4710.
- [13] **Inayat Hussain A.A. (1987):** New Properties of Hypergeometric Series Derivable From Feynman Integrals II. A Generalization of the H-function, *J. Phys. A: Math. Gen.* **20**, pp. 4119-4128.
- [14] **Saxena, V.P. (1982).** Formal Solution of Certain New Pair of Dual Integral Equations Involving H-Functions. *Proc. Nat. Acad. Sci. India Sect A* **51**, pp. 366-375.
- [15] **Srivastava, H.M., Gupta, K.C. and Goyal, S.P. (1982):** The H-Functions of One and Two Variables with Applications. *South Asian Publishers, New Delhi and Madras.*
- [16] **Saigo, M. (1978):** A Remark On Integral Operators Involving the Gauss Hypergeometric Function. *Math. Rep. Kyushu Univ.*, 11, pp. 135-143.
- [17] **Saxena, R.K. and Kumbhat, R.K. (1974):** A generalization of Kober operators. *Vijnana Parishad Anusandhan Patrika*, 16, pp. 31-36.
- [18] **Prabhakar, T.R. (1971),** A Singular Integral Equation with a Generalized Mittag-Leffler Function in the Kernel. *Yokohama math. J.*, 19, pp. 7-15.
- [19] **Fox, C. (1961):** The G and H-Functions as Symmetrical Fourier Kernels. *Trans. Amer. Math. Soc.* 98, pp. 385-429.
- [20] **Wright, E.M. (1935):** The Asymptotic Expansion of the Generalized Bessel Function. *Proc. London Math. Soc.*, **2(38)**, pp. 257–270.
- [21] **Mittag-Leffler, G.M. (1905):** Sur La Representation Analytique D'une Branche uniforme D'une Function Monogene, *Acta Mat.* **29**, pp. 101-182.

