A note on Linear codes associated to Schubert varieties

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Abstract:

We consider the linear code associated with Schubert sub variety of Grassmannian's .In this review article we have studied the basic notions of Schubert code $C_\alpha(l,m)$. We have discussed the some known results and examples of Schubert code.

Key Words :Linear code ,Grassmannian, MDC, Schubert

1.Introduction:

Let q be the power of fixed prime and l and m be the positive integers with $l \leq m$.Let F_q denote the finite field with q elements and F_q^m be the vector space of dimension m over F_q.Let $\begin{bmatrix} m \\ l \end{bmatrix}_q$denote Gaussian binomial coefficient and $G_{l,m}$ denote the Grassmannian of all $l - planes$ of F_q^m. It is also known that the Grassmannian $G_{l,m}$embeds into projective space $P(F_q)^{\binom{m}{l}-1} = P^{\binom{m}{l}-1}$ Via plucker embedding .The image under this mapping is projective algebraic variety .To every projective space one can associate a linear code (example in[1]). The linear code corresponding to $G_{l,m}$ is known as Grassmannian code and it is denoted by $C(l,m)$.The Grassmannian code were introduced by C. T. Ryan in [2,3] for binary case .D Yu Nogin in [4] studied the linear code $C(l,m)$associated to the Grassmannian $G_{l,m}$ over finite field and verified that $C(l,m)$is an $[n,k,d]_q$ code where

$$n = \begin{bmatrix} m \\ l \end{bmatrix}_q = \frac{(q^m-1)(q^m-q)......(q^m-q^{l-1})}{(q^l-1)(q^l-q)......(q^l-q^{l-1})} ,k = \binom{m}{l} \text{ and } d = q^{l(m-l)}$$

For fix integers k,n with $1 \leq k \leq n$ and prime q .Let C be linear $[n,k]_q$code i.e C Is k-dimensional subspace of F_q^n. Given for any $x = (x_1, x_2, ... x_n) \in F_q^n$,define $\sup(x) = \{i: x_i \neq 0\}$ and $\|x\| =$

$|\sup(x)|$ denote support and Hamming norm of x, likewise for $D \subseteq F_q^n$ $\sup(D) = \{i: x_i \neq 0 \; for\ some\ x = (x_1, x_2, \ldots . x_n) \in D\}$ and $\|D\| = |\sup(D)|$denote support and hamming norm of D The minimum distance or hamming weight of C is defined by $d(C) \coloneqq min\{\|x\|: x \in C \; with \; x \neq 0\}$.

For some positive integer r ,the r[th] higher weight or r[th] generalized hamming weight is denoted by $d_r = d_r(C)$ of the code C defined by

$d_r(C) = \min\{\|D\|: D \; is\ subpace\ of\ C\ and \dim(D) = r\}$

The set $\{d_r(C) : 1 \leq r \leq k\}$ is complete weight hierarchy of the code C .For linear code C it is very interesting and difficult to determine complete weight hierarchy .

Let $\alpha = (\alpha_1, \; \alpha_2, \ldots, \alpha_l)$ be strictly increasing sequence of positive integers satisfying

$\alpha_1 \leq \; \alpha_2 \leq \cdots . \leq \alpha_l \leq m$ and $\Omega_\alpha(l, m)$ be corresponding Schubert variety in Grassmannian $G_{l,m}$ Schubert varieties are sub-varieties of Grassmannian .Likewise Grassmannian varieties ,the Schubert variety may considered as subset of $P^{\binom{m}{l}-1}$ The linear code corresponding to Schubert variety $\Omega_\alpha(l, m)$ is called Schubert code and this code is denoted by $C_\alpha(l, m)$.Ghorpade-Tsfasman [5] had proved the minimum distance conjecture for Schubert code corresponding to case when $\delta(\alpha) = l(m-l) - 1$ and found length and dimension od Schubert code in general .In [5] Ghorpade -Tsfasman proved that Schubert code $C_\alpha(l, m)$ is $[n_\alpha, k_\alpha]$ code where $n_\alpha = \sum_{\beta \leq \alpha} q^{\delta(\beta)}$, $k_\alpha = \underset{1 \leq i,j \leq l}{det} \left(\binom{\alpha_j - j + 1}{i - j + 1} \right)$ where sum is run over all $l-$ tuples $\beta = (\beta_1, \beta_2, \ldots \beta_l)$ of integers satisfying $1 \leq \beta_1 \leq \beta_2 \leq \cdots \leq \beta_l \leq m$ and $\beta_i \leq \alpha_i$ for $i = 1,2,3 \ldots l$ and $\delta(\beta) = \sum_{i=1}^{l} (\beta_i - i)$.Alternating proof of MDC for Schubert code was given in [6] and [7].

2. Preliminaries :

The Grassmannian of all l-planes of F_q^m is given by ,

$G_{l,m} \coloneqq \left\{ L \subseteq F_q^m : \text{L is subspace of } F_q^m \text{ and dimL} = l \right\}$.Due to plucker embedding of $G_{l,m}$ into projective space $P^{\binom{m}{l}-1}$,Choose matrix A_L whose rows forms basis for L. The order of A_L will be $l \times m$

matrix with rank l . In A_L we have $\binom{m}{l}$ minors of order l.By fixing some ordering of these minors and map L onto $\binom{m}{l}$ tuples of minors of $A_L$of size l which is required embedding of $G_{l,m}$ into projective space $P^{\binom{m}{l}-1}$.

Let $X = (X_{ij})$ be $l \times m$ indeterminate matrix over F_q.Let l-multiset $I \subset \{1,2, \dots m\}$ denote $l \times l$ minor of X corresponding to the columns indexed by I by $\det_I(X)$ and $F_q[X]_l$ be a vector space over F_q spanning by minors of $\det_I(X)$.Then for any L in $G_{l,m}$,the $l \times m$ matrix A_L is matrix whose rows span L .

Consider the evaluation map $Ev : F_q[X]_l \longrightarrow F_q^{\binom{m}{l}_q}$ defined by

$f = \sum_I \alpha_I det_I(X) \longrightarrow (f(A_L))_{L \in G_{l,m}}$ and $f(A_L) = \sum_I \alpha_I det_I(A_L)$ i.e. $f(A_L)$is evaluation of f at A_L The image of this is called Grassmann code $C(l,m)$.The image $Ev(f)$ of f denoted by c_f is codeword corresponding to f .

Let $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_l)$ be the sequence of strictly increasing positive integers and $A_1 \subset A_2 \dots \dots \subset A_l$ be the sequence of subspaces of F_q^m with $\dim A_i = \alpha_i \ \forall i$.

Let $\Omega_\alpha(l,m) = \{W \in G_{l,m} : \dim(W \cap A_i) \geq i , \forall i\}$ be the Schubert variety in $G_{l,m}$ corresponding to sequence α. If the above evaluation map is restricted to $\Omega_\alpha(l,m)$then it will be Schubert code $C_\alpha(l,m)$ .The Schubert variety $\Omega_\alpha(l,m)$ only depends on the sequence α not on corresponding sequence $A_1 \subset A_2 \dots \dots \subset A_l$ of subspaces .If $B_1 \subset B_2 \dots \dots \subset B_l$ is another sequence of subspaces of F_q^m with $\dim B_i = \alpha_i$ for every i .If $\Omega_\alpha(A_1, A_2, \dots A_l, m)$denotes the Schubert varieties corresponding to the sequence of subspaces $A_1 \subset A_2 \dots \dots \subset A_l$ and $\Omega_\alpha(B_1, B_2, \dots B_l, m)$ denotes Schubert variety corresponding to sequence of subspaces $B_1 \subset B_2 \dots \dots \subset B_l$then due to [1] these corresponding Schubert codes are equivalent.

3. Basic Notions and some known facts about Schubert Code :

For fix integer l, m with $1 \leq l \leq m$.Let $I(l,\ m)$ be the indexing set with partial order $\leq$ for any $\beta = (\beta_1 , \beta_2, \dots \beta_l) \in I(l , m)$.

Let $\delta_\beta \coloneqq \sum_m^l (\beta_i - i) = \beta_1 + \beta_2 + \cdots + \beta_l - \frac{l(l+1)}{2}$

For fix some $\alpha \in I(l,m)$ then $C_\alpha(l,m)$ be the corresponding linear code and length of Schubert code [9] is n_α given by

Thm 1 (Length of Schubert Code)[9]The number of F_q rational points of Ω_α which also length n_α of $C_\alpha(l,m)$ is given by $n_\alpha = \sum_{\beta \le \alpha} q^{\delta_\beta}$ where the sum is over all $\beta \in I(l,m)$ satisfying $\beta \le \alpha$

With the previous notations the explicit formula for the dimension of Schubert code $C_\alpha(l,m)$ is K_α is given by [9] in following result .

Thm 2(Dimension of Schubert Code)

The dimension of Schubert code $C_\alpha(l,m)$ is given by $l \times l$ determinant and given due to [9] by

$$K_\alpha = \underset{1 \le i,j \le l}{det} \left(\binom{\alpha_j - j + 1}{i - j + 1} \right) = \begin{vmatrix} \binom{\alpha_1}{1} & 1 & 0 & & \dots & \mathbf{0} \\ \binom{\alpha_1}{2} & \binom{\alpha_2 - 1}{1} & 1 & & \dots & \mathbf{0} \\ \vdots & & \vdots & & & \vdots \\ \binom{\alpha_1}{l} & \binom{\alpha_2 - 1}{l-1} & \binom{\alpha_3 - 2}{l-2} & \dots & & \binom{\alpha_l - l + 1}{1} \end{vmatrix}$$

Thm 3[9]: The dimension k_α of q-ary Schubert code $C_\alpha(l,m)$ is independent of q and is related to the length $n_\alpha = n_\alpha(q)$ of $C_\alpha(l,m)$ by the formula $\lim_{q \to 1} n_\alpha(q) = k_\alpha$

4.Examples of Schubert Code :

Example 4.1

Let $\alpha = (2,4)$ then $C_\alpha(2,4)$ is corresponding Schubert code then dimension of $C_\alpha(2,4)$ *is* 5

Example 4.2

Let $\alpha = (3,5)$ then $C_\alpha(3,5)$ is corresponding Schubert code and dimension of $C_\alpha(3,5)$ *is* 9

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5. References :

[1]M.Tafsaman ,S Vladut and Nogin Algebraic Geometric Codes :Basic notions Math.Surv Monger ,139(2007)Amer. Math. soc. Providence

[2]Ryan C T , An application of Grassman varieties to coding theory ,Congr. Numer 57(1987)257-271.

[3] Ryan C T ,Projective codes based on Grassmann varieties congr. Numer 57 (273-279)

[4]D Yu Nogin ,Codes associated to Grassmannians Arithmatic, Geometry and Coding Thneory (Luminy,1993), R Pellikaan,M Perret ,S G Vladut,EDS,Walter de Gruyter,Berlin (1996),145-154.

[5] S R Ghorpade and M A Tsfasman Schubert Varieties linear codes and enumerative combinatorics ,Finite fields and their applications,11(2005) 684-699.

[6]X Xaing On the minimum distance conjecture for Schubert Codes,IEE Trans. Inf Theory 54(2008)486-488

[7]S R Ghorpade and P Singh ,Minimum Distance and minimum weight codewords of Schubert Codes , Finite Fields and applications 49(2018), 1-18.

[8]V K Wei ,Generalized hamming weight for linear codes IEEE Trans. Infor Theory ,37 1991 1412-1418

[9]S R Ghorpade ,M A Tsfasman ,Basic Parameters of Schubert Code ,Preprint (2002)

[10]L Guerra ,Rita Vincenti ,on Linear code arising from Schubert varieties ,designs Codes and Cryptography (2004)33:173

[11]S R Ghorpade ,Prasant Singh Minimum Distance and the minimum weight codewords of Schubert Codes,Finite fields and their applications 49.10.1016/jffa.2017.08.014.