COUNTABLY QUASI-BIPARACOMPACT SPACES

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ABSTRACT : This paper is devoted to the development of a covering theory for the bitopological space (X, u, v) based on the notion of a dual cover, which is defined to be a binary relation on the non-empty subsets of X satisfying certain natural conditions. It is shown that under a suitable separation axiom a biparacompact space is fully binormal, but that the converse is false in general. Weakening the local finiteness condition also leads to the consideration of quasi-biparacompactness, etc. Following the present paper on countably quasi-biparacompact spaces the notion of sequential normality is introduced as a weakening of full binormality.

Keywords : countably quasi-biparacompact, quasi-strongly point finite, countably medial.

Introduction

The theory of covers of topological spaces has undergone a rapid development over the past few years, following the pioneering work of Stone [1] and others. The establishment of a similar theory for bitopological spaces faces at the outset the question of deciding on a suitable counterpart to the notion of cover. Indeed it would appear that no one analogue of this notion is entirely satisfactory for all purposes. Pairwise open and weakly pairwise open covers have been the analogue most extensively considered in the literature to date, as witness for instance the papers of Fletcher et.al. [2], Richardson [3], Civic [4] and Datta [5]. Countably paracompact topological spaces were introduced by Dowker in [6]. In this paper we consider some properties of the corresponding class of countably quasi-biparacompact bitopologica1 spaces.

Definition 1. (X, u, v) is countably quasi-biparacompact if every countab1e open dual cover has a quasi-locally finite refinement.

Our principal result is based on the following:

Lemma 1. Let (X, u, v) be a pairwise normal bitopological space, and $d = \{(U_n, V_n) \mid n \in N\}$ an open dual cover satisfying $Un \subseteq U_{n+1}$ and $Vn \subseteq V_{n+1}$ for all $n \in N$. Suppose there is a closed dual cover $c = \{(A_n, B_n) \mid n \in N\}$ with $An \subseteq U_n$ and $Bn \subseteq V_n$ for each n. Then d has a quasi-locally finite countable open refinement.

Proof. Since (X, u, v) is pairwise normal we have for $n \in N$, $s = 1, 2, ...$, sets $R_{ns} \in u$ and $S_{ns} \in v$ with

$A_n \subseteq R_{ns} \in v\text{-cl}[R_{ns}] \subseteq R_{n(s+1)} \subseteq U_n$, and

$B_n \subseteq S_{ns} \in u\text{-cl}[R_{ns}] \subseteq S_{n(s+1)} \subseteq V_n$.

Moreover we may suppose without loss of generality that $R_{ns} \subseteq R_{n(s+1)s}$ and $S_{ns} \subseteq S_{n(s+1)s}$ for if this is not so we may replace R_{ns} and S_{ns} for $n > 0$ by $\bigcup\{R_{ks} \mid k = 0,1,..., n\}$ and $\bigcup\{S_{ks} \mid k = 0,1,..., n\}$ respectively. Let us set:

$$W_{Os} = R_{Os}, \qquad s = 1,2,...;$$

$$W_{ns} = R_{ns} - (u - cl[S_{(n-1)s}]), n = 1,2,..., s = 1,2,...,$$

and $T_{ns} = T_{ns} - (v - cl[R_{(n-1)s}]),\ n = 1,2,..., s = 1,2,...,$

Let us also set

$$R_n = \bigcup\{R_{ns} \mid s = 1,2,...\},$$
$$S_n = \bigcup\{S_{ns} \mid s = 1,2,...\},$$
$$W_n = \bigcup\{W_{ns} \mid s = n, n+1,...\} \text{ and}$$
$$T_n = \bigcup\{T_{ns} \mid s = n, n+1,...\}.$$

Then $e = \{(W_n, S_n) \mid n \in N\} \bigcup \{(R_n, T_n) \mid n \in N\}$

is a countable open quasi-locally finite refinement of d. That e is countable and open is clear; and $e \prec d$ since $W_n \subseteq R_n \subseteq U_n$ and $T_n \subseteq S_n \subseteq V_n$ for each n. To see that it is a dual cover take $x \in X$ and define

$$m(x) = \min\{n \mid \exists s, x \in R_{ns}\},$$

$$n(x) = \min\{n \mid \exists t, x \in S_{nt}\}.$$

Then it is clear that if $m(x) \le n(x)$ we have $x \in W_n \bigcap S_n$ for $n = n(x)$, while if $n(x) \le m(x)$ then $x \in R_n \bigcap T_n$ for $n = m(x)$. Finally to show e is quasi-locally finite take $x \in X$ and suppose that, say, $m(x) \le n(x)$. Then $x \in R_{m(x)s} \subseteq R_{n(x)s}$ for some s; while $x \in S_{n(x)t}$ for some t so we may define:

$$s(x) = \min\{s \mid x \in R_{n(x)s}\},$$
$$t(x) = \min\{t \mid x \in S_{n(x)t}\}.$$

and associate with x the u-nhd. $R_{n(x)s(x)}$ and the v-nhd. $S_{n(x)t(x)}$. It is easy to verify that if $R_{n(x)s(x)} \bigcap T_n \neq \phi$ and $S_{n(x)t(x)} \bigcap R_n \neq \phi$; or if $R_{n(x)s(x)} \bigcap S_n \neq \phi$ and $S_{n(x)t(x)} \bigcap W_n \neq \phi$; then $n \le \max(n(x), s(x))$ or $n \le \max(n(x), t(x))$ respectively. A similar appropriate assignment of nhds. to x may be made when $n(x) < m(x)$. Hence e is quasi-locally finite, and the proof is complete.

In order to state our next theorem we shall need some more terminology. We shall say the dual family d is quasi-strongly point finite if given $x \in X$ either $\{U \mid n \in U \in \text{dom } d\}$ is finite or $\{V \mid n \in V \in \text{ran } d\}$ is finite.

The dual cover d is called countably medial if it can be indexed over IN in such a way that for each $x \in X$ we have $k(x) = \max(m(x), n(x))$, where $m(x) = \min\{n \mid x \in U_n\}$, $n(x) = \min\{n \mid x \in V_n\}$, and $k(x) = \min\{n \mid x \in U_n \cap V_n\}$.

We may now state:

Theorem 1. Let (X, u, v) be a point finitely binormal space satisfying:

(a) Every countable open dual cover has a quasi-strongly point finite open refinement, and

(b) Every countable open dual cover has a countably medial open refinement.

Then (X, u, v) is countably quasi-biparacompact.

Proof. Let d' be a countable open cover of X. By (b) there will be no loss of generality if we assume that d' is countably medial, that is $d' = \{(U'_n V'_n) | n \in N\}$ where $k'(x) = \max(m'(x), n'(x))$ for all $x \in X$, using an obvious notation. Let us set

$$U_n = \bigcup\{(U'_k) | k = 0,1,\ldots,n\} \text{ and } V_n = \bigcup\{(V'_k) | k = 0,1,\ldots,n\}$$

so that $d = \{(U_n, V_n) | n \in N\}$ is an open dual cover satisfying $U_n \subseteq U_{n+1}$ and $V_n \subseteq V_{n+1}$. Let $e = \{(R_\alpha, S_\alpha) | \alpha \in A\}$ be a faithfully indexed quasi-strongly point finite open refinement of d.

For $s \in N$, let

$A(s) = \{\alpha \mid \alpha \in A, R_\alpha \subseteq U_s \text{ and } S_\alpha \subseteq V_s\}$.

Clearly $A(s) \subseteq A(s + 1)$ for all s.

Let $r = \min\{s | A(s) \neq \phi\}$, $R^*_r = \bigcup\{R_\alpha | \alpha \in A(k)\}$ and $S^*_r = \bigcup\{S_\alpha | \alpha \in A(k)\}$. Generally for $s = 1, 2, \ldots$, let

$$R^*_{r+s} = \begin{cases} R^*_{s+r} \text{ if } A(r+s-1) = A(r+s) \\ \bigcup\{R_\alpha | \alpha \in A(r+s) = A(r+s-1)\} \quad \text{otherwise,} \end{cases}$$

$$S^*_{r+s} = \begin{cases} S^*_{s+r} \text{ if } A(r+s-1) = A(r+s) \\ \bigcup\{S_\alpha | \alpha \in A(r+s) = A(r+s-1)\} \quad \text{otherwise.} \end{cases}$$

For $x \in X$, let $s(x) = \min\{s | \exists \alpha \in A(r+s) \text{ with } x \in R_\alpha \bigcap S_\alpha\}$.

Then clearly $x \in R^*_{r+s(x)} \bigcap S^*_{r+s(x)}$ and so $f = \{R^*_{s+r}, S^*_{s+r} | s \in N\}$ is an open dual cover refinement of d.

Let us show it is point finite. For $x \in X$, let $\{\alpha_1, \ldots, \alpha_m\}$ denote the set $\{\alpha | x \in R_\alpha\}$ whenever this set is finite, and otherwise let it denote the set $\{\alpha | x \in S_\alpha\}$. Define

$$p(x) = \begin{cases} 0 \text{ if } \{\alpha_1,\ldots,\alpha_2\} \subseteq A(r) \text{ and otherwise,} \\ \max\{p | \exists i, 1 \le i \le m \text{ with } \alpha_1 \in A(r+p) - A(r+p-1)\}. \end{cases}$$

If, from some point onwards, the sets $A(s)$ are equal then f is finite and hence point finite. In the contrary case, for each $x \in X$,

$$q(x) = \max\{q | A(r + p(x)) - A(r+q)\}$$

is a well defined natural number, and it is clear from the definitions that $x \in R^*_{r+s} \bigcap S^*_{r+s}$ implies $s \le q(x)$. Thus f is point finite as stated. Since (X, u, v) is point finitely binormal. Hence there is an open dual cover $g = \{(M_s, N_s) | s \in N'\}$ where $N' \subseteq N$, $v - cl[M_s] \subseteq R^*_{r+s} \subseteq U^*_{r+s}$ and $u - cl[N_s] \subseteq S^*_{r+s} \subseteq V^*_{r+s}$ for all $s \in N'$.

Let $r' = \min\{s | N'\}$, and $t = r + r'$. Put $A_t = v - cl[M_{r'}]$, $B_t = u - cl[N_{r'}]$, and generally for $s = 1, 2, \ldots$,

$$A_{t+s} = \begin{cases} v - \text{cl}[M_{r'+s}] \text{ if } r'+s \in N' \\ A_{t+s-1} \quad \text{otherwise,} \end{cases}$$

$$B_{t+s} = \begin{cases} u - \text{cl}[N_{r'+s}] \text{ if } r'+s \in N' \\ B_{t+s-1} \quad \text{otherwise,} \end{cases}$$

Then $c = \{(A_n, B_n) | n = t, t+1, \ldots\}$ is a closed dual cover, $A_n \subseteq U_n$ and $B_n \subseteq V_n$. It follows that the conditions of Lemma 1 are satisfied for the open dual cover $d_t = \{(U_n, V_n) | n = t, t+1, \ldots\}$, **,** and so we have an open quasi-locally finite refinement

$$e_t = \{(W_n, S_n) | n = t, t+1, \ldots\} \bigcup \{(R_n, T_n) | n = t, t+1, \ldots\}.$$

For $n \in N$, set $W'_n = (\bigcup\{W_k | k = n \vee t, n \wedge t+1, \ldots\} \bigcap U'_n$,

$S'_n = (\bigcup\{S_k | k = n \vee t, n \wedge t+1, \ldots\} \bigcap V'_n$,

$R'_n = (\bigcup\{R_k | k = n \vee t, n \wedge t+1, \ldots\} \bigcap U'_n$ and

$T'_n = (\bigcup\{T_k | k = n \vee t, n \wedge t+1, \ldots\} \bigcap V'_n$.

If $x \in W_n \bigcap S_n$ or $x \in R_n \bigcap T_n$ then $n \ge k'(x) = \max(m'(x), n'(x))$, and so $x \in W'_{k'(x)} \bigcap S'_{k'(x)}$ or $x \in R'_{k'(x)} \bigcap T'_{k'(x)}$ respectively. This shows that

$$e' \in \{W'_n, S'_n | W'_n \bigcap S'_n \neq \phi\} \bigcup \{R'_n, T'_n | R'_n \bigcap T'_n \neq \phi\}$$

is an open dual cover refiniment of d'. Finally the argument used in the proof of Lemma 1 to show e_t is quasi-locally finite will also show that e' is quasi-locally finite, and the proof is complete.

The next result is also a consequence of Lemma 1.

Proposition 1. Let (X, u, v) be a pairwise perfectly normal space [7], and suppose that each countable open dual cover has a countably medial open refinement. Then (X, u, v) is countably quasi-biparacompact.

Proof. Let $d' = \{(U'_n, V'_n) \mid n \in N\}$ be a countably medial open dual cover, and form $d = \{(U_n, V_n) \mid n \in N\}$ with $U_n \subseteq U_{n+1}$, $V_n \subseteq V_{n+1}$ as in the proof of Theorem 1. Now we have v-closed sets P_{ns}, $s \in N$, and u-closed sets Q_{ns}, $s \in N$, so that $P_{ns} \subseteq P_{n(s+1)}$, $Q_{ns} \subseteq Q_{n(s+1)}$, $U_n = \bigcup\{P_{ns} | s \in N\}$, and $V_n \bigcup\{Q_{ns} | s \in N\}$.

For $n \in N$ define $A_n = \bigcup\{P_{tn} | t = 1,\ldots,n\} \subseteq U_n$ and $B_n = \bigcup\{Q_{ns} | t = 1,\ldots,n\} \subseteq V_n$. Then

$c = \{(A_n, B_n) \mid n \in N\}$

is a closed dual cover, and the conditions of Lemma 1 are satisfied. The remainder of the proof is similar to the last part of the proof of Theorem 1, and is ommited.

The final lemma of this paper deals with a situation at the opposite extreme from that of Lemma 1. This result can also be useful in establishing (countable) quasi-biparacompactness in some cases.

Lemma 2. Let (X, u, v) be a pairwise normal bitopological space $c = \{(A_n, B_n) \mid n \in N\}$. If $d = \{(U_n, V_n) \mid k \in Z\}$ is a countable open dual cover satisfying $\bigcap\{U_k\} = \bigcap\{V_k\} = \phi, U_k \subseteq U_{k+1}$ and $V_k \subseteq V_{k+1}$ for all $k \in Z$, and if there exists a closed dual cover $c = \{(A_k, B_k) \mid k \in Z\}$ with $A_k \subseteq A_{k+1}, B_{k+1} \subseteq B_k, A_k \subseteq U_k$ and $B_k \subseteq V_k$ for all $k \in Z$, then d has a quasi-locally finite countable open refinement.

Proof. Since (X, u, v) is pairwise normal we have u-open sets R_k with $A_k \subseteq R_k \subseteq v - \mathrm{cl}\{R_k\} \subseteq U_k$. Without loss of generality we may also suppose that $R_k \subseteq R_{k+1}$ for each $k \in Z$, for if this is not so we may replace R_k by $\bigcup\{R_i \mid i = 1, ..., k\}$ for $k > 0$, and by $\bigcap\{R_i \mid i = k, ..., 0\}$ for $k < 0$. In just the same way we have v-open sets S_k with $B_k \subseteq S_k \subseteq u - \mathrm{cl}\{S_k\} \subseteq V_k$, and we may suppose $S_{k+1} \subseteq S_k$ for all $k \in Z$.

Clearly $e = \{(R_k, S_k) \mid k \in Z\}$ is an open refinement of d. We show it is quasi-locally finite. For $x \in X$ the numbers

$$m(x) = \min\{k \mid x \in v - \mathrm{cl}[R_k]\},$$

$$n(x) = \max\{k \mid x \in u - \mathrm{cl}[S_k]\},$$

both exist in Z. Also, for some k', $x \in R_k, \bigcap S_k$, and so $m(x) \leq k' \leq n(x)$ for each $x \in X$. Now

$$M(x) = R_{n(x)}(u - \mathrm{cl}[S_{n(x)+1}]\})$$

is a u-nhd of x, and

$$N(x) = S_{m(x)}(v - \mathrm{cl}[R_{m(x)+1}]\})$$

is a v-nhd of x. Also if $M(x) \bigcap S_k \neq \phi$ and $N(x) \bigcap R_k \neq \phi$ then $m(x) \leq k \leq n(x)$. Hence e is quasi-locally finite, and the proof is complete.

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