

Some Fixed Point Theorems in Digital Metric Spaces Satisfying Certain Rational Inequalities

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Abstract : In this paper we have proved some results in Digital Metric Space which have been established in the framework of other metric spaces.

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I. INTRODUCTION

Digital image processing is one of the most flourishing area of engineering research. It has tremendous potential of utilizing available computing power with help of optimized algorithms and produce wonderful results of much importance. It was Rosenfeld[5] in 1979 who coined the term Digital Topology. He discussed it in a paper titled “Digital Topology”, which then became the founding literature of Digital Topology. It discusses the topological aspects of the images. The geometry and topology of images helped a lot in the field of image processing and pattern recognition.

Another area of much development is fixed point theory. It has evolved to a great extent that it has persuaded many mathematicians of 20th century invest their time in finding fruitful results from it. Nonlinear analysis is considered to be the source of this theory. Fixed point theory has helped finding solutions to differential equations, integral equations and equations representing dynamical systems. In 1922, Banach[3] gave the contraction principle which ensures the existence and uniqueness of the fixed point of certain self-maps. The theorem states that,

Let (X, d) be a non-empty complete metric space with a self-map $S : X \rightarrow X$ satisfying

$$d(Sx, Sy) \leq \alpha d(x, y)$$

For all $x, y \in X$ and $0 \leq \alpha < 1$. Then S has a unique fixed-point u in X .

In this paper we establish a theorem using some rational inequality and give some corollaries which are direct consequence of our theorem.

II. PRELIMINARIES

We now discuss the basics of the digital metric space from the topological point of view. Adjacency and neighbourhood are discussed next to give fundamental background on the topic. The fundamental object in digital topology is a Lattice, which is used to represent a digital image in n -dimensions. The lattice has lattice points (with integer coordinates) which are called **pixels(2d)** or **voxels(3d)**. A **Digital Plane \mathbb{Z}^2** is a set of all the points in \mathbb{R}^2 and 3-D digital space **\mathbb{Z}^3** is a set of all the points in \mathbb{R}^3 having integer coordinates.



Figure 1. 2D and 3D digital planes

Definition 2.1 Let l, n be positive integers with $1 \leq l \leq n$. Consider two distinct points $p = \{p_1, p_2, \dots, p_n\}, q = \{q_1, q_2, \dots, q_n\} \in \mathbb{Z}^n$. The points p and q are k_l -adjacent if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices $j, |p_j - q_j| \neq 1, p_j = q_j$.

2.1.1 (2-adjacency) Two points on digital plane are said to be 2-adjacent if $|p - q| = 1$.

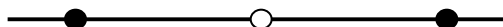


Figure 2. 2 – adjacency

2.1.2 (4 – neighbours) The 4 – neighbours of a point p_{ij} are its four horizontal and vertical neighbours $(i \pm 1, j)$ and $(i, j \pm 1)$. 4 – neighbours of a point p_{ij} are denoted by $N_4(p_{ij})$.

2.1.3 (6 – neighbours) The 6 – neighbours of a point p_{ij} are its four horizontal and vertical neighbours $(i \pm 1, j)$ and $(i, j \pm 1)$ along with 2 neighbours $(i + 1, j + 1)$ and $(i - 1, j - 1)$ or $(i - 1, j + 1)$ and $(i + 1, j - 1)$ i.e. from either of the diagonals. 6 – neighbours of a point p_{ij} are denoted by $N_6(p_{ij})$.

2.1.4 (8 – neighbours) The 8 – neighbours of a point p_{ij} consist of its 4 – neighbours together with its four diagonal neighbours $(i + 1, j \pm 1)$ and $(i - 1, j \pm 1)$ and are denoted by $N_8(p_{ij})$. The diagonal neighbours of a point p_{ij} are denoted by $N_D(p_{ij})$. The 4 – neighbours, $N_4(p_{ij})$ and 4 diagonal neighbours, $N_D(p_{ij})$ are together called as 8 – neighbours of the point p_{ij} and are denoted by $N_8(p_{ij})$.

2.1.5 (4 – adjacency) Two points p and q on a digital plane (\mathbb{Z}^2) are said to be 4-adjacent if $q \in N_4(p_{ij})$.

2.1.6 (6 – adjacency) Two points p and q on a digital plane (\mathbb{Z}^2) are said to be 6-adjacent if $q \in N_6(p_{ij})$.

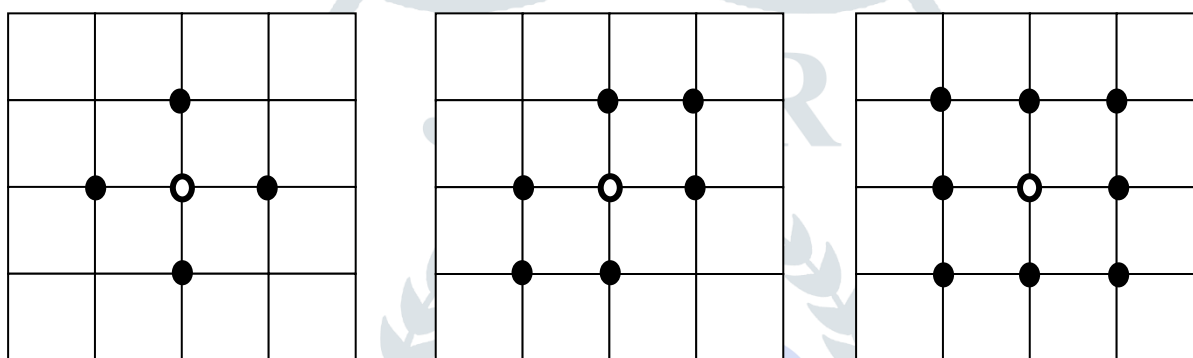


Figure 3. 4 – adjacency(left), 6 – adjacency(center) 8 – adjacency(right)

2.1.7 (8 – adjacency) Two points p and q on a digital plane (\mathbb{Z}^2) are said to be 8-adjacent if $q \in N_8(p_{ij})$.

2.1.8 (6 – adjacency in \mathbb{Z}^3) Two points p and q are 6 – adjacent in 3-D digital space (\mathbb{Z}^3) if point q is located at coordinates $(i \pm 1, j, k)$, $(i, j \pm 1, k)$ or $(i, j, k \pm 1)$ from the point p_{ijk} .

2.1.9 (18 – adjacency in \mathbb{Z}^3) Two points p and q are 18 – adjacent in a 3-D digital space (\mathbb{Z}^3) if point q is located at coordinates $(i \pm 1, j \pm 1, k)$, $(i \pm 1, j \mp 1, k)$, $(i \pm 1, j, k \pm 1)$, $(i \pm 1, j, k \mp 1)$, $(i, j \pm 1, k \pm 1)$ or $(i, j \pm 1, k \mp 1)$ from the point p_{ijk} .

2.1.10 (26 – adjacency in \mathbb{Z}^3) Two points p and q are 26 – adjacent in a 3-D digital space (\mathbb{Z}^3) if point q is located at coordinates $(i \pm 1, j \pm 1, k \pm 1)$, $(i \pm 1, j \pm 1, k \mp 1)$, $(i \pm 1, j \mp 1, k \pm 1)$ or $(i \mp 1, j \pm 1, k \pm 1)$ from the point p_{ijk} .

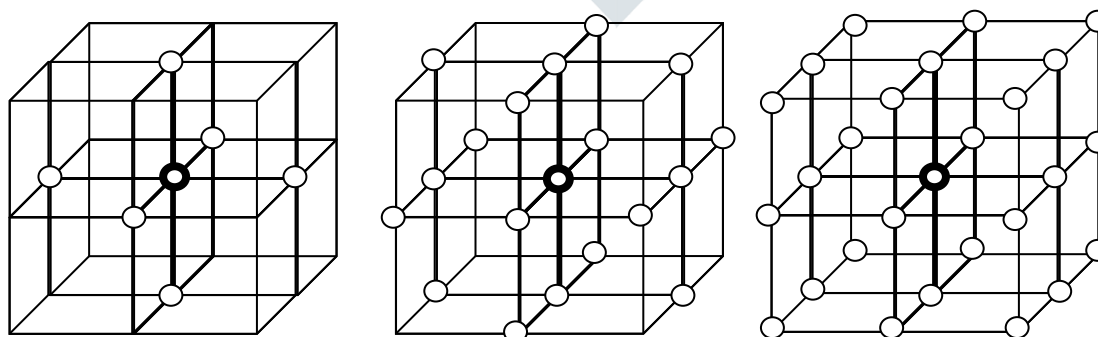


Figure 4. 6(left), 18(center) & 26 – adjacency(right)

Definition 2.2. A digital image is a pair (X, κ) , where $\Phi \neq X \subset \mathbb{Z}^n$ for some positive integer n and κ is an adjacency relation on X . Technically, a digital image (X, κ) is an undirected graph whose vertex set is the set of members of X and whose edge set is the set of unordered pairs $\{x_0, x_1\} \subset X$ such that $x_0 \neq x_1$ and x_0 and x_1 are κ – adjacent.

Definition 2.3. Let $(X, \kappa) \subset \mathbb{Z}^{n_0}$ and $(Y, \kappa_1) \subset \mathbb{Z}^{n_1}$ be digital images and $f: X \rightarrow Y$ be a function.

- If for every κ_0 - connected subset U of X , $f(U)$ is a κ_1 - connected subset of Y then f is said to be (κ_0, κ_1) - continuous.
- f is (κ_0, κ_1) - continuous if for every κ_0 - adjacent points $\{x_0, x_1\}$ of X , either $f(x_0) = f(x_1)$ or $f(x_0)$ and $f(x_1)$ are κ_1 - adjacent in Y .
- If f is (κ_0, κ_1) - continuous, bijective and f^{-1} is (κ_0, κ_1) - continuous, then f is called (κ_0, κ_1) - isomorphism and denoted by $X_{(\kappa_0, \kappa_1)} \cong Y$.

Now we present basic terminology required for further discussion. Let (X, d, κ) be a digital metric space with κ - adjacency and Euclidean metric d for \mathbb{Z}^n .

Definition 2.4. [2] Let (X, κ) be a digital image set. Let d be a function from $(X, \kappa) \times (X, \kappa) \rightarrow \mathbb{Z}^n$ satisfying all the properties of metric space. The triplet (X, d, κ) is called a digital space.

Proposition 2.5. [4] A sequence $\{x_n\}$ of points of a digital metric space (X, d, κ) is a *Cauchy sequence* if and only if there is $\alpha \in \mathbb{N}$ such that $d(x_n, x_m) < \alpha, \forall n, m > \alpha$.

Theorem 2.6. [4] For a digital space (X, d, κ) , if a sequence $\{x_n\} \subset X \subset \mathbb{Z}^n$ is a *Cauchy sequence* then there is $\alpha \in \mathbb{N}$ such that we have $x_n = x_m, \forall n, m > \alpha$.

Proposition 2.7. [4] A sequence $\{x_n\}$ of points of a digital metric space (X, d, κ) converges to a limit $l \in X$ if for all $\epsilon > 0$, there is $\alpha \in \mathbb{N}$ such that $d(x_n, l) < \epsilon, \forall n > \alpha$.

Proposition 2.8. [4] A sequence $\{x_n\}$ of points of digital metric space (X, d, κ) converges to a limit $l \in X$ if there is $\alpha \in \mathbb{N}$ such that $x_n = l, \forall n > \alpha$.

Theorem 2.9. [3] A digital metric space (X, d, κ) is complete.

Definition 2.10. [2] Let (X, d, κ) be any digital metric space. A self-map f on a digital metric space is said to be a digital contraction if there exists a $\lambda \in [0, 1)$ such that for all $x, y \in X$,

$$d(f(x), f(y)) \leq \lambda d(x, y).$$

Proposition 2.11. [2] Every digital contraction map $f: (X, d, \kappa) \rightarrow (X, d, \kappa)$ is digitally continuous.

Proposition 2.12. [4] In a digital metric space (X, d, κ) , consider two points x_i, x_j in a sequence $\{x_n\} \subset X$ such that they are κ - adjacent. Then they have the Euclidean distance $d(x_i, x_j)$ which is greater than or equal to 1 and at most \sqrt{t} depending on the position of the two points.

Theorem 2.13 [2] Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation between the objects of X . Let (X, d, κ) be a digital metric space and S be a self-map on X satisfying the following:

$$d(Sx, Sy) \leq \alpha \{d(x, Sx) + d(y, Sy)\}$$

For all $x, y \in X$ and $0 < \alpha < \frac{1}{2}$. Then S has a unique common fixed point in X .

Theorem 2.14 [2] Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation in X . Let (X, d, κ) be a digital metric space and S be a self map on X satisfying the following:

$$d(Sx, Sy) \leq \alpha \{d(x, Sy) + d(y, Sx)\}$$

For all $x, y \in X$ and $0 < \alpha < \frac{1}{2}$. Then S has a unique common fixed point in X .

Theorem 2.15 [2] Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation in X . Let (X, d, κ) be a digital metric space and S be a self map on X satisfying the following:

$$d(Sx, Sy) \leq \alpha d(x, Sx) + \beta d(y, Sy) + \gamma d(x, y)$$

For all $x, y \in X$ and all nonnegative real numbers α, β, γ with $(\alpha + \beta + \gamma) < 1$. Then S has a unique common fixed point in X .

III. MAIN RESULTS

Theorem 3.1. Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation in X . Let (X, d, κ) be a digital metric space and S, T be self maps on X satisfying the following:

$$\begin{aligned} d(S(x, y), T(u, v)) &\leq \alpha [d(x, u) + d(y, v)] + \beta [d(x, S(x, y)) + d(u, T(u, v))] \\ &+ \gamma [d(x, T(u, v)) + d(u, S(x, y))] + \delta \left[\frac{d(x, S(x, y))d(u, T(u, v))}{[d(x, u) + d(y, v)]} \right] \\ &+ \eta \left[\frac{[d(x, u) + d(y, v)][d(x, S(x, y)) + d(u, T(u, v))]}{[1 + d(x, u) + d(y, v)]} \right] \\ &+ \zeta \left[\frac{[d(u, S(x, y)) + d(x, T(u, v))]}{[1 + d(u, T(u, v))d(u, S(x, y))]} \right] \end{aligned}$$

For all u, v, x and $y \in X$ and $2(\alpha + \beta + \eta) + 4(\gamma + \zeta) + \delta < 1$. Then S & T have a unique common fixed point in X .

Proof: Let $x_0, y_0 \in X$ be some arbitrary in X . Define a sequence $\langle x_n \rangle$ and $\langle y_n \rangle$ in X such that $x_{n+1} = S(x_n, y_n)$, $y_{n+1} = T(y_n, x_n)$, and $x_{n+2} = T(x_{n+1}, y_{n+1})$, $y_{n+2} = S(y_{n+1}, x_{n+1})$ for $n \in \mathbb{N}$.

Consider

$$d(x_n, x_{n+1}) = d(S(x_{n-1}, y_{n-1}), T(x_n, y_n))$$

$$\begin{aligned}
 d(x_n, x_{n+1}) &\leq \alpha[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] + \beta[d(x_{n-1}, S(x_{n-1}, y_{n-1})) + d(x_n, T(x_n, y_n))] \\
 &+ \gamma[d(x_{n-1}, T(x_n, y_n)) + d(x_n, S(x_{n-1}, y_{n-1}))] + \delta \left[\frac{[d(x_{n-1}, S(x_{n-1}, y_{n-1}))d(x_n, T(x_n, y_n))]}{[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]} \right] \\
 &+ \frac{\eta \left[[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)][d(x_{n-1}, (x_{n-1}, y_{n-1})) + d(x_n, T(x_n, y_n))] \right]}{[1 + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]} \\
 &+ \zeta \left[\frac{[d(x_n, S(x_{n-1}, y_{n-1})) + d(x_{n-1}, T(x_n, y_n))]}{[1 + d(x_n, T(x_n, y_n))d(x_n, S(x_{n-1}, y_{n-1}))]} \right]
 \end{aligned}$$

Now, from the assumptions of the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ we have,

$$\begin{aligned}
 d(x_n, x_{n+1}) &\leq \alpha[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
 &+ \gamma[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] + \delta \left[\frac{[d(x_{n-1}, x_n)d(x_n, x_{n+1})]}{[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]} \right] \\
 &+ \frac{\eta \left[[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)][d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \right]}{[1 + d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]} \\
 &+ \zeta \left[\frac{[d(x_n, x_n) + d(x_{n-1}, x_{n+1})]}{[1 + d(x_n, x_{n+1})d(x_n, x_n)]} \right]
 \end{aligned}$$

Now, if a & $b \in \mathbb{R}$ and $a < b$ with $a > 0$, then following relation holds,

$$\frac{ab}{1+a} < b \tag{1}$$

Therefore, from (1), we can rewrite the inequality as,

$$\begin{aligned}
 d(x_n, x_{n+1}) &\leq \alpha[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] + \beta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
 &+ \gamma[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] + \delta d(x_n, x_{n+1}) + \eta[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\
 &+ \zeta[d(x_n, x_n) + d(x_{n-1}, x_{n+1})]
 \end{aligned}$$

$$\begin{aligned}
 d(x_n, x_{n+1}) &\leq \alpha d(x_{n-1}, x_n) + \alpha d(y_{n-1}, y_n) + \beta d(x_{n-1}, x_n) + \beta d(x_n, x_{n+1}) \\
 &+ \gamma[d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + d(x_n, x_{n+1}) + d(x_{n+1}, x_n)] + \delta d(x_n, x_{n+1}) \\
 &+ \eta d(x_{n-1}, x_n) + \eta d(x_n, x_{n+1}) \\
 &+ \zeta[d(x_n, x_{n+1}) + d(x_{n+1}, x_n) + d(x_{n-1}, x_n) + d(x_n, x_{n+1})]
 \end{aligned}$$

$$\begin{aligned}
 d(x_n, x_{n+1}) &\leq d(x_{n-1}, x_n)(\alpha + \beta + \gamma + \eta + \zeta) + \alpha d(y_{n-1}, y_n) \\
 &+ d(x_n, x_{n+1})(\beta + 2\gamma + \delta + \eta + 2\zeta) + d(x_{n+1}, x_n)(\gamma + \zeta)
 \end{aligned}$$

Since d is a measure of Euclidean distance, we can rewrite $d(x_{n+1}, x_n)$ as $d(x_n, x_{n+1})$ therefore, we have,

$$\begin{aligned}
 d(x_n, x_{n+1}) &\leq d(x_{n-1}, x_n)(\alpha + \beta + \gamma + \eta + \zeta) + \alpha d(y_{n-1}, y_n) \\
 &+ d(x_n, x_{n+1})(\beta + 3\gamma + \delta + \eta + 3\zeta)
 \end{aligned}$$

$$\begin{aligned}
 d(x_n, x_{n+1})[1 - (\beta + 3\gamma + \delta + \eta + 3\zeta)] &\leq d(x_{n-1}, x_n)(\alpha + \beta + \gamma + \eta + \zeta) \\
 &+ \alpha d(y_{n-1}, y_n)
 \end{aligned}$$

$$\begin{aligned}
 d(x_n, x_{n+1}) &\leq d(x_{n-1}, x_n) \left[\frac{(\alpha + \beta + \gamma + \eta + \zeta)}{[1 - (\beta + 3\gamma + \delta + \eta + 3\zeta)]} \right] \\
 &+ d(y_{n-1}, y_n) \left[\frac{\alpha}{[1 - (\beta + 3\gamma + \delta + \eta + 3\zeta)]} \right]
 \end{aligned}$$

Similarly, it can be shown that,

$$\begin{aligned}
 d(y_n, y_{n+1}) &\leq d(y_{n-1}, y_n) \left[\frac{(\alpha + \beta + \gamma + \eta + \zeta)}{[1 - (\beta + 3\gamma + \delta + \eta + 3\zeta)]} \right] \\
 &+ d(x_{n-1}, x_n) \left[\frac{\alpha}{[1 - (\beta + 3\gamma + \delta + \eta + 3\zeta)]} \right]
 \end{aligned}$$

Now we add the two expressions,

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq [d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] \left[\frac{(\alpha + \beta + \gamma + \eta + \zeta)}{[1 - (\beta + 3\gamma + \delta + \eta + 3\zeta)]} \right]$$

Now, let $\rho = \left[\frac{(\alpha + \beta + \gamma + \eta + \zeta)}{[1 - (\beta + 3\gamma + \delta + \eta + 3\zeta)]} \right]$

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq \rho[d(x_{n-1}, x_n) + d(y_{n-1}, y_n)]$$

Since $(\alpha + \beta + 3\gamma + \delta + \eta + 3\zeta) < 1$, it is implied that $\rho < 1$. Now, continuing like this we get,

$$d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \leq \rho^n[d(x_0, x_1) + d(y_0, y_1)]$$

Now we take $n \rightarrow \infty$, we get

$$[d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \rightarrow 0$$

This also implies that $d(x_n, x_{n+1}) \rightarrow 0$ and $d(y_n, y_{n+1}) \rightarrow 0$. Therefore, the sequences $\langle x_n \rangle$ and $\langle y_n \rangle$ are Cauchy sequences in digital metric space. Thus, there exist points u and $v \in X$ such that $\lim_{n \rightarrow \infty} x_n = u$ and $\lim_{n \rightarrow \infty} y_n = v$. Since

S and T are continuous and $x_{n+1} = S(x_n, y_n)$, $y_{n+1} = T(y_n, x_n)$, and $x_{n+2} = T(x_{n+1}, y_{n+1})$, $y_{n+2} = S(y_{n+1}, x_{n+1})$. Now as $n \rightarrow \infty$, we have,

$$\lim_{n \rightarrow \infty} S(x_n, y_n) = \lim_{n \rightarrow \infty} x_{n+1} \\ S(u, v) = u$$

In a similar way,

$$\lim_{n \rightarrow \infty} T(y_n, x_n) = \lim_{n \rightarrow \infty} y_{n+1} \\ S(v, u) = v$$

This implies that $u, v \in X$. Thus there exists a coupled fixed point in digital metric space. Now we prove the uniqueness of the fixed point.

$$\begin{aligned} d(u, v) &= d(S(u, v), T(u, v)) \\ d(S(u, v), T(u, v)) &\leq \alpha[d(u, u) + d(v, v)] + \beta[d(u, S(u, v)) + d(u, T(u, v))] \\ &\quad + \gamma[d(u, T(u, v)) + d(u, S(u, v))] + \delta \left[\frac{[d(u, S(u, v))d(u, T(u, v))]}{[d(u, u) + d(v, u)]} \right] \\ &\quad + \eta \left[\frac{[d(u, u) + d(v, u)][d(u, S(u, v)) + d(u, T(u, v))]}{[1 + d(u, u) + d(v, v)]} \right] \\ &\quad + \zeta \left[\frac{[d(u, S(u, v)) + d(u, T(u, v))]}{[1 + d(u, T(u, v))d(u, S(u, v))]} \right] \\ d(S(u, v), T(u, v)) &\leq \alpha[d(u, u) + d(v, v)] + \beta[d(u, u) + d(u, u)] \\ &\quad + \gamma[d(u, u) + d(u, u)] + \delta \left[\frac{[d(u, u)d(u, u)]}{[d(u, u) + d(v, u)]} \right] \\ &\quad + \eta \left[\frac{[d(u, u) + d(v, u)][d(u, u) + d(u, u)]}{[1 + d(u, u) + d(v, v)]} \right] \\ &\quad + \zeta \left[\frac{[d(u, u) + d(u, u)]}{[1 + d(u, u)d(u, u)]} \right] \end{aligned}$$

From (1) we have,

$$d(u, v) = d(S(u, v), T(u, v)) \leq (2\alpha + 2\beta + 2\gamma + \delta + 2\eta + \zeta)d(u, u)$$

Also, we can have,

$$d(v, u) = d(S(v, u), T(v, u)) \leq (2\alpha + 2\beta + 2\gamma + \delta + 2\eta + \zeta)d(v, u)$$

Adding these two we yield,

$$d(u, v) + d(v, u) \leq (4\alpha + 4\beta + 4\gamma + 2\delta + 4\eta + 2\zeta)[d(u, u) + d(v, u)]$$

Since, $(4\alpha + 4\beta + 4\gamma + 2\delta + 4\eta + 2\zeta) < 1$, the inequality is only possible when,
 $d(u, v) + d(v, u) = 0$

This implies that,

$$d(u, v) = d(v, u) = 0$$

Now consider,

$$\begin{aligned} d(w, z) &= d(S(w, z), T(w, z)) \\ d(S(w, z), T(w, z)) &\leq \alpha[d(w, w) + d(z, z)] + \beta[d(w, S(w, z)) + d(w, T(w, z))] \\ &\quad + \gamma[d(w, T(w, z)) + d(w, S(w, z))] + \delta \left[\frac{[d(w, S(w, z))d(w, T(w, z))]}{[d(w, w) + d(z, w)]} \right] \\ &\quad + \eta \left[\frac{[d(w, w) + d(z, w)][d(w, S(w, z)) + d(z, T(w, z))]}{[1 + d(w, w) + d(z, z)]} \right] \\ &\quad + \zeta \left[\frac{[d(w, S(w, z)) + d(w, T(w, z))]}{[1 + d(w, T(w, z))d(w, S(w, z))]} \right] \\ d(S(w, z), T(w, z)) &\leq \alpha[d(w, w) + d(z, z)] + \beta[d(w, w) + d(w, w)] \\ &\quad + \gamma[d(w, w) + d(w, w)] + \delta \left[\frac{[d(w, w)d(w, w)]}{[d(w, w) + d(z, w)]} \right] \end{aligned}$$

$$\begin{aligned}
& +\eta \left[\frac{[d(w, w) + d(z, w)][d(w, w) + d(w, w)]}{[1 + d(w, w) + d(z, z)]} \right] \\
& +\zeta \left[\frac{[d(w, w) + d(w, w)]}{[1 + d(w, w)d(w, w)]} \right]
\end{aligned}$$

From (1) we have,

$$d(w, z) = d(S(w, z), T(w, z)) \leq (2\alpha + 2\beta + 2\gamma + \delta + 2\eta + \zeta)d(w, w)$$

Also, we can have,

$$d(z, w) = d(S(v, u), T(v, u)) \leq (2\alpha + 2\beta + 2\gamma + \delta + 2\eta + \zeta)d(z, w)$$

Adding these two we yield,

$$d(w, z) + d(z, w) \leq (4\alpha + 4\beta + 4\gamma + 2\delta + 4\eta + 2\zeta)[d(w, w) + d(z, w)]$$

Since, $(4\alpha + 4\beta + 4\gamma + 2\delta + 4\eta + 2\zeta) < 1$, the inequality is only possible when,

$$d(w, z) + d(z, w) = 0$$

This implies that,

$$d(w, z) = d(z, w) = 0$$

This proves the theorem.

Corollary 3.2 Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation in X . Let (X, d, κ) be a digital metric space and S, T be self maps on X satisfying the following:

$$\begin{aligned}
d(S(x, y), T(u, v)) & \leq \alpha[d(x, u) + d(y, v)] + \beta[d(x, S(x, y)) + d(u, T(u, v))] \\
& +\gamma[d(x, T(u, v)) + d(u, S(x, y))] + \delta \left[\frac{[d(x, S(x, y))d(u, T(u, v))]}{[d(x, u) + d(y, u)]} \right] \\
& +\eta \left[\frac{[d(x, u) + d(y, u)][d(x, S(x, y)) + d(u, T(u, v))]}{[1 + d(x, u) + d(y, v)]} \right]
\end{aligned}$$

For all u, v, x and $y \in X$ and $[2(\alpha + \beta + \eta) + 4\gamma + \delta] < 1$. Then S & T have a unique common fixed point in X .

Corollary 3.3 Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation in X . Let (X, d, κ) be a digital metric space and S, T be self maps on X satisfying the following:

$$\begin{aligned}
d(S(x, y), T(u, v)) & \leq \alpha[d(x, u) + d(y, v)] + \beta[d(x, S(x, y)) + d(u, T(u, v))] \\
& +\gamma[d(x, T(u, v)) + d(u, S(x, y))] + \delta \left[\frac{[d(x, S(x, y))d(u, T(u, v))]}{[d(x, u) + d(y, u)]} \right]
\end{aligned}$$

For all u, v, x and $y \in X$ and $[2(\alpha + \beta) + 4\gamma + \delta] < 1$. Then S & T have a unique common fixed point in X .

Corollary 3.4 Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation in X . Let (X, d, κ) be a digital metric space and S, T be self maps on X satisfying the following:

$$\begin{aligned}
d(S(x, y), T(u, v)) & \leq \alpha[d(x, u) + d(y, v)] + \beta[d(x, S(x, y)) + d(u, T(u, v))] \\
& +\gamma[d(x, T(u, v)) + d(u, S(x, y))]
\end{aligned}$$

For all u, v, x and $y \in X$ and $[2(\alpha + \beta) + 4\gamma] < 1$. Then S & T have a unique common fixed point in X .

Corollary 3.5 Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation in X . Let (X, d, κ) be a digital metric space and S, T be self maps on X satisfying the following:

$$d(S(x, y), T(u, v)) \leq \alpha[d(x, u) + d(y, v)] + \beta[d(x, S(x, y)) + d(u, T(u, v))]$$

For all u, v, x and $y \in X$ and $[2(\alpha + \beta)] < 1$. Then S & T have a unique common fixed point in X .

Corollary 3.6 Let (X, κ) be a digital image where $X \subset \mathbb{Z}^n$ and κ is an adjacency relation in X . Let (X, d, κ) be a digital metric space and S, T be self maps on X satisfying the following:

$$d(S(x, y), T(u, v)) \leq \alpha[d(x, u) + d(y, v)]$$

For all u, v, x and $y \in X$ and $2\alpha < 1$. Then S & T have a unique common fixed point in X .

IV. CONCLUSION

In this paper we have proved some theorems in the context of digital metric space which extended the work done in framework of digital metric space. We also established some corollaries which are direct consequence of our theorem 3.1. It also worth noting that there exists a coupled fixed point in Digital Metric Space.

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