SOME GENERALIZED PROBLEMS IN THERMOELASTICITY

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***Abstract*—This work is devoted to a study of the induced temperature and stress fields in an elastic half space in context of classical coupled thermo elasticity and generalized thermo elasticity in a unified system of equations. The half space is considered to be made of an isotropic homogeneous thermo elastic material. The bounding plane surface is heated by a non-Gaussian laser beam with pulse duration of 2 ps. An exact solution of the problem is first obtained in Laplace transform space. Since the response is of more interest in the transient state, the inversion of Laplace transforms have been carried numerically. The derived expressions are computed numerically for copper and the results are presented in graphical form.**

IndexTerms—Thermo elasticity; Coupled Thermo elasticity; Generalized Thermo elasticity; Non-Gaussian Laser Pulse.

I. INTRODUCTION

Although thermo mechanical phenomena in the majority of practical engineering applications are adequately simulated with the classical Fourier heat conduction equation, there is an important body of problems that require due consideration of thermo mechanical coupling: it is appropriate in these cases to apply the generalized theory of thermo elasticity. Serious attention has been paid to the generalized thermo elasticity theories in solving thermo elastic problems in place of the classical uncoupled/coupled theory of thermo elasticity.

The absence of any elasticity term in the heat conduction equation for uncoupled thermo elasticity appears to be unrealistic, since due to the mechanical loading of an elastic body, the strain so produced causes variation in the temperature field. Moreover, the parabolic type of the heat conduction equation results in an infinite velocity of thermal wave propagation, which also contradicts the actual physical phenomena. Introducing the strain-rate term in the uncoupled heat conduction equation, Biot extended the analysis to incorporate coupled thermo elasticity [1]. In this way, although the first shortcoming was over, there remained the parabolic type partial differential equation of heat conduction, which leads to the paradox of infinite velocity of the thermal wave. To eliminate this paradox generalized thermo elasticity theory was developed subsequently. Due to the advance particle accelerators, etc. which can supply heat pulses with a very fast time-rise [2, 3]; generalized thermo elasticity theory is receiving serious attention. The development of the second sound effect has been nicely reviewed by Chandrasekharaiah [4]. At present mainly two different models of generalized thermo elasticity are being extensively used-one proposed by Lord and Shulman [5] and the other proposed by Green and Lindsay [6]. L-S (Lord and Shulman theory) suggests one relaxation time and according to this theory, only Fourier's heat conduction equation is modified; while G-L (Green and Lindsay theory) suggests two relaxation times and both the energy equation and the equation of motion are modified.

The so-called ultra-short lasers are those with pulse duration ranging from nanoseconds to femtoseconds in general. In the case of ultra-short-pulsed laser heating, the high-intensity energy flux and ultra-short duration laser beam, have introduced situations where very large thermal gradients or an ultra-high heating speed may exist on the boundaries [7]. In such cases, as pointed out by many investigators, the classical Fourier model, which leads to an infinite propagation speed of the thermal energy, is no longer valid [8]. The non-Fourier effect of heat conduction takes into account the effect of mean free time (thermal relaxation time) in the energy carrier's collision process, which can eliminate this contradiction. By employing the L-S model (Lord and Shulman) with one relaxation time, Sherief and Anwar [9] have obtained the distributions of thermal stresses and temperature for a generalized thermo elastic problem in which an infinite elastic space was subjected to the influence of a continuous line source of heat. The solution of the problem was obtained by applying the Hankel and Laplace integral transforms successively. Wang and Xu have studied the stress wave induced by nanoseconds, picoseconds, and femtoseconds laser pulses in a semi-infinite solid [10]. The solution takes into account the non-Fourier effect in heat conduction and the coupling effect between temperature and strain rate. It is known that characteristic elastic waveforms are generated when a pulsed laser irradiates a metal surface. Point in case, McDonald has studied the importance of thermal diffusion to the thermo elastic wave generation [11]. Bagri and Eslami got the unified generalized thermo elasticity solution for cylinders and spheres [12].

The present investigation is devoted to a study of the induced temperature and stress fields in an elastic half space under the purview of classical coupled thermo elasticity and generalized thermo elasticity in a unified sys- tem of field equations. The half space continuum is considered to be made of an isotropic homogeneous thermo elastic material, the bounding plane surface being subjected to a Non-Gaussian laser pulse. An exact solution of the problem is first obtained in Laplace transform space. Since the response is of more interest in the transient state, the inversion of Laplace transforms have been carried numerically. The derived expressions are computed numerically for copper and the results are presented in graphical form.

II. BASIC EQUATIONS AND FORMULATION

All the field equations represented by (CTE), (L-S) and (G-L) can be formulated in the following unified system [13] and [14]:

$$\rho \ddot{u}_i = \mu u_{i,jj} + (\lambda+\mu) u_{j,ji} + F_i - \gamma\left(1+\upsilon\frac{\partial}{\partial t}\right)T_{,i}, \quad (1)$$

which constitute equation of motion where are Lame's constants, i u is the displacement component, F is the body force component, $\gamma = \alpha_T(3\lambda+2\mu)$, and T is the thermal expansion, □ is relaxation time, T is the temperature of the body and

$$KT_{,ii} = \rho C_E\left(\frac{\partial}{\partial t}+\tau_o\frac{\partial^2}{\partial t^2}\right)T + \left(1+n\tau_o\frac{\partial}{\partial t}\right)\left(T_o\gamma\dot{u}_{j,j}-\rho Q\right), \quad (2)$$

which constitute equation of heat conduction where K is the thermal conductivity, CE is the specific heat at constant strain, o is relaxation time, o T is the reference temperature, n is a parameter and Q is the heat source.

$$\sigma_{ij} = \mu\left(u_{i,j}+u_{j,i}\right)+\left[\lambda u_{i,i}-\gamma\left(T+\upsilon\dot{T}\right)\right]\delta_{ij}. \quad (3)$$

which is called constitutive equation where ij σ_{ij} is the stress tensor and ij σ_{ij} is the Kronecker function. Equations (1)-(3) reduce to coupled thermo elasticity (CTE) when $\tau_o = \upsilon = 0$. Putting , $n=1$, $\upsilon=0$ and $\tau_o>0$, the equations reduce to Lord-Shulman (L-S) model, while when $n=0$, $\tau_o>0$ and $\upsilon>0$, the equations reduce to Green-Lindsay (G-L) model [13,14].

III. THE NON-GAUSSIAN LASER PULSE

We will consider the medium is heated uniformly by a laser pulse with non-Gaussian form temporal profile [7].

$$L(t) = \frac{L_0 t}{t_p^2}\exp\left(-\frac{t}{t_p}\right), \quad (4)$$

where $t_p = 2\,\text{ps}$ is a characteristic time of the laser- pulse (the time duration of a laser pulse), L0 is the laser intensity which is defined as the total energy carried by a laser pulse per unit area of the laser beam, see Figure 1, [7].

The conduction heat transfer in the medium can be modeled as a one-dimensional problem with an energy source Q(x,t) near the surface, i,e.

$$Q(x,t) = \frac{1-R}{\delta}\exp\left(\frac{x-h/2}{\delta}\right)I(t) = \frac{R_a L_0}{\delta t_p^2}\, t\,\exp\left(\frac{x-h/2}{\delta}-\frac{t}{t_p}\right), \quad (5)$$

where δ is the absorption depth of heating energy and Ra is the surface reflectivity [7]. When we consider the laser pulse lie on the surface of the medium when x=0 (see Figure 1), we get the energy source in the form

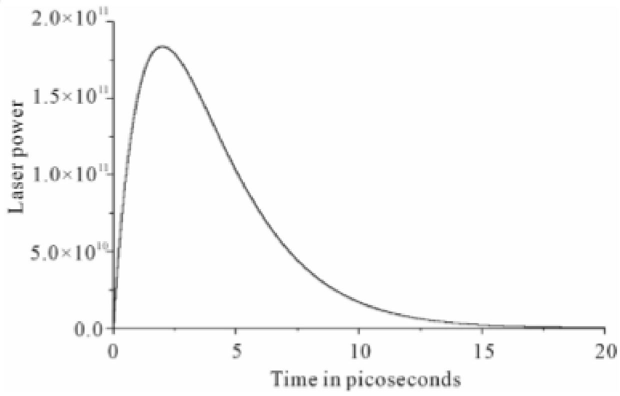

Figure 1. Temporal profile of laser power L/L0

$$Q(t) = \frac{R_a L_0}{\delta t_p^2}\, t\,\exp\left(\frac{-h}{2\delta}-\frac{t}{t_p}\right) \quad (6)$$

I. FORMULATION OF THE PROBLEM

We consider half-space (x>=0) with the x-axis pointing into the medium with initial temperature distribution T. This half-space is irradiated uniformly the bounding plane (x = 0) by a laser pulse with non-Gaussian temporal profile as in (6). We assume that there is no body forces affecting the medium and all the state functions initially are equal to zero.

The displacement vector has the components:

$$u = u(x,t), v = w = 0.$$

Hence, the governing Equations (1)-(3) in one-dimensional will take the following forms:

The equation of motion

$$\rho\ \ddot{u} = (\lambda + 2\mu)\ \frac{\partial^2 u}{\partial x^2} - \gamma \frac{\partial}{\partial x}\left(1 + \upsilon \frac{\partial}{\partial t}\right)\theta\ , \qquad (8)$$

where $\theta = |T - T_0|$ is the temperature increment. The heat equation:

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{\rho C_E}{K}\left[\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right]\theta + \frac{\gamma T_0}{K}\left[\frac{\partial}{\partial t} + n\tau_0 \frac{\partial^2}{\partial t^2}\right]e$$
$$- \left(1 + n\,\tau_0 \frac{\partial}{\partial t}\right)\frac{\rho R_a L_0}{K t_p^2\, \delta}\ t \exp\left(\frac{-h}{2\delta} - \frac{t}{t_p}\right), \qquad (9)$$

Where

$$e = \frac{\partial u}{\partial x}\ . \qquad (10)$$

The constitute equation:

$$\sigma_{xx} = (\lambda + 2\mu)e - \gamma\left(1 + \upsilon \frac{\partial}{\partial t}\right)\theta \qquad (11)$$

For simplicity, we will use the following non-dimensional variables Youssef (2006):

$$(x', u', h', \delta') \equiv c_0 \eta (x, u, h, \delta),$$
$$(t', t_p', \tau_0', \upsilon') \equiv c_0^2 \eta (t, t_p, \tau_0, \upsilon).$$
$$\theta' = \frac{\gamma}{(\lambda + 2\mu)}\theta,\ \sigma_{ij}' = \frac{\sigma_{ij}}{\lambda + 2\mu}. \qquad (12)$$

where $c_o = \sqrt{\frac{\lambda + 2\mu}{\rho}}$ is the longitudinal wave speed

and $\eta = \frac{\rho C_E}{K}$ is the thermal viscosity. Hence, we have the following system of equations (we have dropped the prime for convenient)

$$\ddot{e} = \frac{\partial^2 e}{\partial x^2} - \frac{\partial^2}{\partial x^2}\left(1 + \upsilon \frac{\partial}{\partial t}\right)\theta\ , \qquad (13)$$

$$\frac{\partial^2 \theta}{\partial x^2} = \left[\frac{\partial}{\partial t} + \tau_0 \frac{\partial^2}{\partial t^2}\right]\theta + \varepsilon_1\left[\frac{\partial}{\partial t} + n\tau_0 \frac{\partial^2}{\partial t^2}\right]e$$
$$- \varepsilon_2\left(1 + n\,\tau_0 \frac{\partial}{\partial t}\right)t \exp\left(-\frac{t}{t_p}\right), \qquad (15)$$

$$\sigma_{xx} = e - \left(1 + \upsilon \frac{\partial}{\partial t}\right)\theta\ , \qquad (16)$$

Where $\varepsilon_1 = \frac{\gamma^2 T_0}{\rho C_E (\lambda + 2\mu)}$ is the dimensionless thermo elastic coupling constant, and $\varepsilon_2 = \frac{R_a L_0 \gamma}{t_p^2\, \delta K c_o} \exp\left(\frac{-h}{2\delta}\right)$.

IV. The Exact Solution of the Problem in THE LAPLACE Transform Domain

Applying the Laplace transform for Equations (13)-(15) defined by the formula

$$\bar{f}(s) = L[f(t)] = \int_0^\infty f(t)\, e^{-st}\, dt$$

Hence, we obtain the following system of differential equations

$$\left(D_x^2 - s^2\right)\bar{e} = \left(1+\upsilon s\right) D_x^2 \bar{\theta} , \qquad (17)$$

$$\left[D_x^2 - \left(s + \tau_0 s^2\right)\right]\bar{\theta} = \varepsilon_1\left(s + n\tau_0 s^2\right)\bar{e} - F(s), \qquad (18)$$

$$\bar{\sigma}_{xx} = \bar{e} - \left(1+\upsilon s\right)\bar{\theta} , \qquad (19)$$

where all the state functions initially are equal to zero,

$$D_x^n = \frac{d^n}{dx^n} \quad \text{and} \quad F(s) = \frac{\varepsilon_2\left(1+n\tau_0 s\right)}{\left(s+1/t_p\right)^2} .$$

Eliminating e between the Equations (17) and (18), we get

$$\left[\frac{d^4}{dx^4} - L\frac{d^2}{dx^2} + M\right]\bar{\theta}(x,s) = s^2 F(s) , \qquad (20)$$

Where $L = s^2 + \left(s + \tau_o s^2\right) + \varepsilon_1\left(s + n\tau_o s^2\right)\left(1+\upsilon s\right)$ and $M = s^2\left(s + \tau_o s^2\right)$

The solution of Equation (20) takes the following form

$$\bar{\theta}(x,s) = \frac{F(s)}{s+\tau_o s^2} + \sum_{i=1}^{2} A_i\left(\lambda_i^2 - s^2\right)\exp\left(-\lambda_i x\right) . \qquad (21)$$

where $\pm\lambda_1$ and $\pm\lambda_2$ are the roots of the characteristic equation and

$$\lambda^4 - L\lambda^2 + M = 0 , \qquad (22)$$

To get the value of the parameters A1 and A2 we have to apply the boundary conditions on the bounding plane x=0 of the assumed half space as follows:

$$\bar{e}(x,s) = \left(1+\upsilon s\right)\sum_{i=1}^{2} A_i \lambda_i^2 \exp\left(-\lambda_i x\right) . \qquad (23)$$

which gives after applying Laplace transform

$$\theta(0,t) = e(0,t) = 0 , \qquad (24)$$

After applying the above boundary conditions, we get

$$A_1 = \frac{F(s)\lambda_2^2}{\left(s+\tau_o s^2\right)s^2\left(\lambda_2^2 - \lambda_1^2\right)}$$

V. Numeric Results

$$g(t) = \frac{e^{\kappa t}}{t}\left[\frac{1}{2}\bar{g}(\kappa) + Re\sum_{n=1}^{N}(-1)^n \bar{g}\left(\kappa + \frac{i n\pi}{t}\right)\right], \qquad (30)$$

where Re is the real part and is imaginary number unit. For faster convergence, numerous numerical experiments have shown that the value of satisfies the relation $\kappa t \approx 4.7$ [8].

With a view to illustrating the analytical procedure presented earlier, we now consider a numerical example for which computational results are given. For this purpose, copper is taken as the thermo elastic material, [13]:

$$K = 386\ \mathrm{kg\cdot m\cdot k^{-1}\cdot s^{-3}},\ \alpha_T = 1.78\,(10)^{-5}\ \mathrm{k^{-1}},\ T_o = 293\mathrm{k}$$
$$\rho = 8954\ \mathrm{kg\cdot m^{-3}},\ C_E = 383.1\ \mathrm{m^2\cdot k^{-1}\cdot s^{-2}},$$
$$L_0 = 1\times10^{11}\ \mathrm{J/m^2},\ \mu = 3.86\times10^{10}\ \mathrm{kg\cdot m^{-1}\cdot s^{-2}},$$
$$\lambda = 7.76\times10^{10}\mathrm{kg\cdot m^{-1}\cdot s^{-2}},\ R_a = 0.5,\quad h = 0.1,\quad \delta = 0.01,$$
$$t_p = 2.0,\quad \tau_o = 0.02,\quad \upsilon = 0.08,\quad \varepsilon_1 = 0.0168,$$
$$\varepsilon_2 = 2.1\times10^4.$$

The computations were carried out for t = 0.2 and the temperature, the stress, the strain and the displacement distributions are represented graphically at different positions of x

The **Figures 2-5** show that, the laser pulse makes the difference between the results in the context of the three studied models CTE, L-S and G-L is very clear and we can differentiate between them, while it was very difficult previously when we used thermal loading by using thermal shock or ramp-type heating as in [13,14].

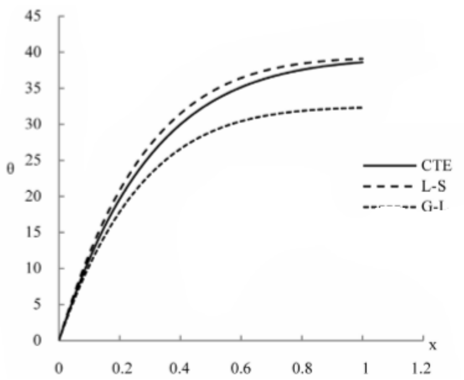

Figure 2. The temeperature distribution.

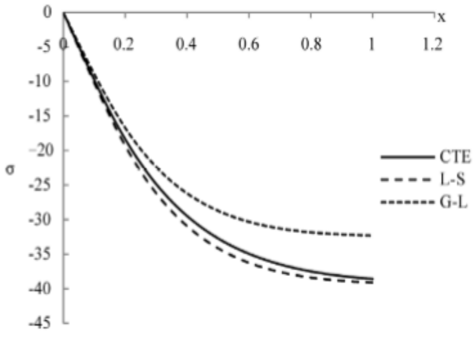

Figure 3. The stress distribution.

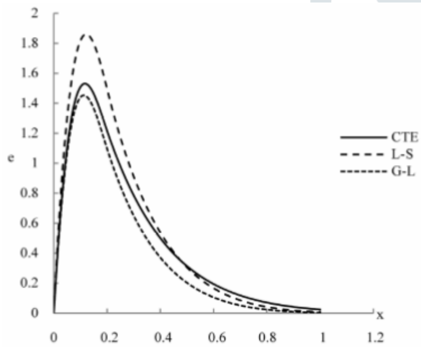

Figure 4. The strain distribution.

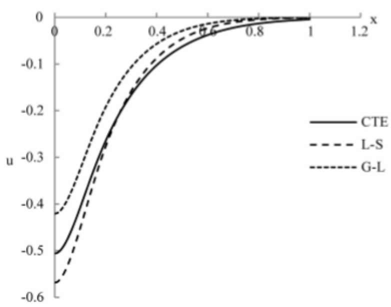

Figure 5. The displacement distribution.

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