0-DISTRIBUTIVE ALMOST SEMILATTICES

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Abstract: The concept of 0-distributive almost semilattice is introduced and proved some basic properties of 0-distributive almost semilattice. A set of equivalent conditions for an almost semilattice (ASL) with 0 to become 0-distributive ASL is established. Also, obtained another set of equivalent condition for an ASL with 0, in which intersection of any family of S-ideals of L is again an S-ideal to become 0-distributive ASL. We gave a necessary and sufficient conditions for a proper filter of an ASL L to become a maximal filter and proved that every maximal filter in 0-distributive ASL is a prime filter. Also, we proved that if L is 0-distributive ASL with unimaximal element, in which intersection of any family of S-ideals is again an S-ideal, then for any filter F of L and for any annihilator ideal I of L such that $F \cap I = \phi$, there exists a prime filter of L containing F and disjoint with I. Finally, we introduced the concepts of disjunctive ASL, weakly disjunctive ASL and we gave necessary and sufficient conditions for a 0-distributive ASL to become disjunctive (weakly disjunctive) ASL.

Key Words : 0-distributive ASL, maximal filter, prime filter, disjunctive ASL, weakly disjunctive ASL.

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1.INTRODUCTION

A Boolean lattice is a lattice which is both complemented and distributive. The concepts of distributivity and complementedness in a lattice are in a sense independent. Frink [2] generalized the concept of complementedness in a lattice to that of pseudo-complementedness in a semilattice. Further Nemitz [10] extended the concept of pseudo-complemented semilattice to that of implicative semilattice. On the other hand there were attempts to generalize the notion of distributivity. Gratzer and Schmidt [3] introduced the concept of distributivity in a semilattice which was investigated in detail by Rhodes [12]. The generalization of distributivity motivated Balbes [1] to introduce a nice concept of primeness in a semilattice which is invariably a generalization of distributivity in a semilattice. It is well known that the class of 0-distributive semilattices contains pseudo-complemented and implicative semilattices on one hand and distributive and prime semilattices on the other hand. Also, known that the concept of pseudo-complementedness is independent to that of distributivity in a semilattice and one is sweetly puzzeled to find how 0-distributivity includes both pseudo-complementedness and distributivity. 0-distributive were lattices discussed by Varlet [13] and Hofmann-Keimel [4]. Y.S.Pawar and N.K.Thakare [11] extended the concept of 0-distributivity in lattices to semilattices and they proved basic properties of 0-distributive semilattices and established different characterization of 0-distributive semilattices.

In this paper, we introduced the concept of 0-distributive ASL and gave certain examples of 0-distributive ASLs, which are not 0-distributive semilattices. We derive a set of equivalent conditions for an almost semilattice with 0 to become 0-distributive ASL and also derive a set of equivalent conditions for an ASL with 0, in which intersection of any family of S-ideals of L is again an S-ideal to become 0-distributive ASL. We proved necessary and sufficient conditions for a proper filter of an ASL to become a maximal filter and proved that every maximal filter in 0-distributive ASL is a prime filter. Also, we proved that if L is 0-distributive ASL with unimaximal element, in which intersection of any family of S-ideals is again an S-ideal, then for any filter F of L and for any annihilator ideal I of L such that $F \cap I = \phi$, there exists a prime filter of L containing F and disjoint with I. Finally, we introduced the concepts of disjunctive ASL, weakly disjunctive ASL and we gave necessary and sufficient conditions for a 0-distributive ASL to become disjunctive (weakly disjunctive) ASL.

2 Preliminaries

In this section we collect a few important definitions and results which are already known and which will be used more frequently in the text.

Definition 2.1 : A semilattice is an algebra $(S,*)$, where S is a nonempty set and $*$ is a binary operation on S satisfying:

(1) $x*(y*z)=(x*y)*z$

(2) $x*y=y*x$

(3) $x*x=x$

Definition 2.2 : A semilattice S with 0 is called 0-distributive if for any $a,b,c\in S$ such that $a\wedge b=0$, $a\wedge c=0$ implies $a\wedge d=0$ $for some d\geq b,c$.

Definition 2.3 : An ASL with 0 is an algebra $(L,o,0)$ of type (2, 0) satisfies the following conditions:

(1) $(x o y) o z = x o (y o z)$

(2) $(x o y) o z = (y o x) o z$

(3) $x o x = x$

(4) $0 o x = 0$, $for all\, x,y,z\in L$

Definition 2.4 : Let L be an ASL. A nonempty subset I of L is said to be an S-ideal if it satisfies the following conditions:

(1) $If\ x\in I\ and\ a\in L, then\ x o a\in I.$

(2) $If\ x,y\in I, then\ there\ exists\ d\in I\ such\ that\ d o x=x, d o y=y.$

Definition 2.5 : Let L be an ASL and $a\in L$. Then $(a]=\{a o x : x\in L\}$ is an S-ideal of L and is called principal S-ideal generated by a.

Theorem 2.6 : Let L be an ASL. Then the set $PSI(L)$, of all principal S-ideals of L is a semilattice with respect to set intersection.

Definition 2.7 : A nonempty subset F of an ASL L is said to be a filter if F satisfies the following conditions:

(1) $x,y\in F\ implies\ x o y\in F$

(2) $If\ x\in F\ and\ a\in L\ such\ that\ a o x=x, then\ a\in F$

Definition 2.8: Let L be an ASL and $a\in L$. Then $[a)=\{x\in L : x o a=a\}$ is a filter of L and is called principal filter generated by a..

Definition 2.9: Let L be an ASL with 0. Then a unary operation $a\mapsto a^*$ on L is said to be pseudo-complementation on L if for any $a,b\in L$, it satisfies following conditions:

(1) $a o a^* = 0$

(2) $a o b = 0 \Rightarrow a^* o b = b$

Definition 2.10: For any nonempty subset A of an ASL L with 0, define $A^* = \{x\in L : x o a = 0,\ for\ \ all\ a\in L\}$. Then A^* is called the annihilator of A.

Note that if $A=\{a\}$, then we write $[a]^*$ instead of A^*.

Definition 2.11: Let L be an ASL with 0. An element $a\in L$ is said to be dense if $[a]^*=\{0\}$.

Theorem 2.12 : Let L be an ASL with 0. Then for any ideals $I, J\ of\ L$, we have the following:

(1) $I^* = \bigcap_{a\in I}(a]^*$

(2) $(I\cap J)^* = (J\cap I)^*$

(3) $I\subseteq J \Rightarrow J^* \subseteq I^*$

(4) $I^*\cap J^* \subseteq (I\cap J)^*$

(5) $(I\cap J)^{**} = I^{**}\cap J^{**}$

(6) $I\subseteq I^{**}$

(7) $I^{***} = I^*$

(8) $I^*\subseteq J^* \Leftrightarrow J^{**}\subseteq I^{**}$

(9) $I\cap J=(0] \Leftrightarrow I\subseteq J^* \Leftrightarrow J\subseteq I^*$

(10) $(I \cup J)^* = I^* \cap J^*$

Theorem 2.13 : Let L be an ASL with 0. Then for any $x, y \in L,$ we have the following:

(1) $x \le y \Rightarrow [y]^* \subseteq [x]^*$

(2) $[x]^* \subseteq [y]^* \Rightarrow [y]^{**} \subseteq [x]^{**}$

(3) $x \in [x]^{**}$

(4) $(x]^* = [x]^*$

(5) $(x] \cap [x]^* = \{0\}$

(6) $[x o y]^* = [y o x]^*$

(7) $[x]^* \cap [y]^* \subseteq [x o y]^*$

(8) $[x o y]^{**} = [x]^{**} \cap [y]^{**}$

(9) $[x]^{***} = [x]^*$

(10) $[x]^* \subseteq [y]^*$ *if and only if* $[y]^{**} \subseteq [x]^{**}$

Definition 2.14 : Let L and L' be two ASLs with zero elements 0 and $0'$ respectively. Then a mapping $f: L \to L'$ is called a homomorphism if it satisfies the following:

(1) $f(a o b) = f(a) o f(b),$ *for all* $a, b \in L$

(2) $f(0) = 0'$

Definition 2.15 : An element $m \in L$ is said to be unimaximal if $m o x = x$ *for all* $x \in L.$

Theorem 2.16 : Let L be an ASL with unimaximal element. Then the set $F(L),$ of all filters in L form a lattice with respect to set inclusion, where for any $F, G \in F(L), F \wedge G = F \cap G$ *and* $F \vee G = \{t \in L: t o (a o b) = a o b$ *for some* $a \in F, b \in G\}.$

Definition 2.17 : Let L be an ASL with unimaximal element. Then a proper filter F of L is said to be a prime filter if for any filters F_1 *and* F_2 *of* $L, F_1 \cap F_2 \subseteq F$ implies that either $F_1 \subseteq F$ *or* $F_2 \subseteq F.$

3 0-Distributive Almost Semilattices

In this section we introduce the concept of 0-distributive ASL and give certain examples of 0-distributive ASLs, which are not 0-distributive semilattices. We prove some basic properties of 0-distributive ASLs. We derive necessary and sufficient condition for an ASL with 0 to become 0-distributive ASL. Also, we establish a set of equivalent conditions for if L is an ASL with 0, which satisfies intersection of any family of S-ideal is again an S-ideal, then L become a 0-distributive ASL. We give necessary and sufficient condition for a proper filter to become a maximal filter in ASL and prove that every maximal filter is a prime filter in 0-distributive ASL. Also, we prove that, if L is a 0-distributive ASL with unimaximal element in which intersection of any family of S-ideals is again an S-ideal, then for any filter F of L and for any annihilator ideal I of L such that $F \cap I = \phi,$ there exists a prime filter containing F and disjoint with $I.$ Finally, we introduce the concepts of disjunctive and weakly disjunctive ASLs and give necessary and sufficient condition for a 0-distributive ASL to become disjunctive (weakly disjunctive) ASL. First, we observe the following:

Example3.1 : Let $L_1 = \{0, a\}$ *and* $L_2 = \{0, b_1, b_2\}$ be two ASLs. Put $L = L_1 \times L_2 = \{(0,0), (0, b_1),$ $(0, b_2), (a, 0), (a, b_1), (a, b_2)\}.$ Define a binary operation o on L as follows:

o	(0,0)	(0,b_1)	(0,b_2)	(a,0)	(a,b_1)	(a,b_2)
(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)	(0,0)
(0,b_1)	(0,0)	(0,b_1)	(0,b_2)	(0,0)	(0,b_1)	(0,b_2)
(0,b_2)	(0,0)	(0,b_1)	(0,b_2)	(0,0)	(0,b_1)	(0,b_2)
(a,0)	(0,0)	(0,0)	(0,0)	(a,0)	(a,0)	(a,0)
(a,b_1)	(0,0)	(0,b_1)	(0,b_2)	(a,0)	(a,b_1)	(a,b_2)
(a,b_2)	(0,0)	(0,b_1)	(0,b_2)	(a,0)	(a,b_1)	(a,b_2)

Then clearly, L is an ASL with 0. Now, put $L' = \{(0,0),(a,0),(0,b_1),(0,b_2)\}$. Then clearly L' is a sub ASL of an ASL L. In this ASL L', we have $(0,0)o(a,0)=(0,0)$ and $(0,0)o(0,b_1)=(0,0)$. But, no $(x,y)\in L'$ such that $(x,y)o(a,0)=(a,0),(x,y)o(0,b_1)=(0,b_1)$ and $(x,y)o(0,0)=(0,0)$. Now, we introduce the concept of 0-distributive ASL.

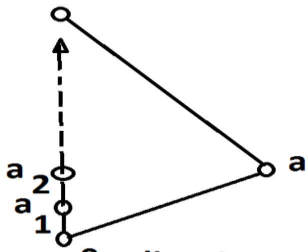
Definition 3.2 : Let L be an ASL with 0. Then L is said to be 0-distributive ASL if for any $x,y,z\in L$, $x\,o\,y=0$ and $x\,o\,z=0$ then there exists $d\in L$ such that $d\,o\,y=y, d\,o\,z=z$ and $d\,o\,x=0$.

It can be easily observe that, in example 3.1, L is a 0-distributive ASL, but not 0-distributive semilattice and also observe that every 0-distributive semilattice is 0-distributive ASL.

Theorem 3.3 : Every pseudo-complemented ASL is 0-distributive ASL.

Proof : Suppose L is a pseudo-complemented ASL. Now, we shall prove that L is 0-distributive ASL. Let $x,y,z\in L$ such that $x\,o\,y=0$ and $x\,o\,z=0$. Then we have $x^*\,o\,y=y, x^*\,o\,z=z$ and $x\,o\,x^*=0$. Now, put $d=x^*$. Then $d\,o\,y=y, d\,o\,z=z$ and $d\,o\,x=0$. Therefore L is 0-distributive ASL.

But, the converse of the above theorem is not true. For, consider the following ASL, whose Hasse diagram is as follows:



dia - 1

Clearly, this is 0-distributive ASL, but, not pseudo-complemented, since a^* does not exist.

In the following we give some examples of 0-distributive ASLs.

Example 3.4: Let $L=\{0,a,b,c\}$ and define a binary operation o on L as follows:

o	0	a	b	c
0	0	0	0	0
a	0	a	a	a
b	0	a	b	c
c	0	a	b	c

Then clearly $(L,o,0)$ is an ASL with 0 and also, clearly L is a 0-distributive ASL, but, not 0-distributive semilattice.

Example 3.5 : Let $L=\{0,a,b,c\}$ and define a binary operation o on L as follows:

o	0	a	b	c
0	0	0	0	0
a	0	a	b	c
b	0	a	b	c
c	0	c	c	c

Then clearly $(L, o, 0)$ is an ASL with 0 and also, clearly L is a 0-distributive ASL, but, not 0-distributive semilattice.

In general, $[a]^*$ is not an S-ideal in an ASL L, for every $a \in L$. For, consider the following example.

Example 3.6 : Let $L = \{0, a, b, c, d\}$ and define a binary operation o on L as follows:

o	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
d	0	a	b	c	d

Then clearly, (L, o) is an ASL with 0. Now, put $S = \{0, a, b, c\}$. Then clearly, S is a sub ASL of an ASL L. In this ASL S, $[a]^* = \{0, b, c\}$. Now, we have $b, c \in [a]^*$, but, we have no $t \in [a]^*$ such that $t o b = b and t o c = c$. Hence $[a]^*$ is not an S-ideal. However, if $[a]^*$ is an S-ideal for all a in an ASL L, then we have the following.

Theorem 3.7 : Let L be an ASL with 0. Then L is 0-distributive if and only if $[a]^*$ is an S-ideal, for all $a \in L$.

Proof : Suppose L is 0-distributive ASL and $a \in L$. Now, we shall prove that $[a]^*$ is an S-ideal. Let $x \in [a]^*$ $and t \in L$. Then $x o a = 0$. It follows that $(x o t) o a = 0$. Thus $x o t \in [a]^*$. Let $x, y \in [a]^*$. Then $x o a = 0$, $y o a = 0$. Since L is 0-distributive, there exists $t \in L$ $such that t o x = x$, $t o y = y$ and $t o a = 0$ and hence $t \in [a]^*$. Thus $[a]^*$ is an S-ideal. Conversely, suppose $[a]^*$ is an S-ideal, for all $a \in L$. Now, we shall prove that L is 0-distributive ASL. Let $x, y, z \in L$ such that $x o y = 0$ and $x o z = 0$. Then $y \in [x]^*$ $and z \in [x]^*$. Since $[x]^*$ is an S-ideal, there exists $d \in [x]^*$ and hence $x o d = 0$ such that $d o y = y, d o z = z$. Thus L is 0-distributive ASL.

In the following, we give necessary and sufficient condition for an ASL L with 0 to become 0-distributive ASL in terms of principal S-ideals.

Theorem 3.8 : Let L be an ASL with 0. Then L is 0-distributive if and only if the semilattice $PSI(L)$ is 0-distributive semilattice.

Proof : Suppose L is 0-distributive ASL. Now, we shall prove that $PSI(L)$ is 0-distributive semilattice. Let $(x], (y], (z] \in PSI(L)$ $such that$ $(x] \cap (y] = (0]$ and $(x] \cap (z] = (0]$. Then $(x o y] = (0]$ and $(x o z] = (0]$. Threfore $x o y = 0$ and $x o z = 0$. Hence there exists $d \in L$ $such$ $that$ $d o y = y$, $d o z = z$ and $d o x = 0$. It follows that $(d] \cap (y] = (d o y] = (y]$, $(d] \cap (z] = (d o z] = (z]$ and $(d o x] = (0]$. Hence $(y] \subseteq (d], (z] \subseteq (d]$ and $(d] \cap (x] = (0]$. Thus $PSI(L)$ is 0-distributive semilattice. Conversely, suppose $PSI(L)$ is 0-distributive semilattice. Now, we shall prove that L is 0-distributive ASL. Let $x, y, z \in L$ such that $x o y = 0, x o z = 0$. Then $(x] \cap (y] = (x o y] = (0]$, $(x] \cap (z] = (x o z] = (0]$. Therefore there exist $(d] \in PSI(L)$ such that $(y] \subseteq (d], (z] \subseteq (d]$ and $(x] \cap (d] = (0]$. It follows that $(d] \cap (y] = (y]$, $(d] \cap (z] = (z]$ and $(d] \cap (x] = (0]$. Hence $(d o y] = (y], (d o z] = (z]$ and $(d o x] = (0]$. Now, we have $y \in (y] = (d o y]$. Therefore $y = (d o y) o y = d o (y o y) = d o y$. Similarly, we get $d o z = z$. Clearly, $d o x = 0$. Thus L is 0-distributive ASL.

Corollary 3.9 : Let L be an ASL with 0. Then the following are equivalent:

(1) L is 0-distributive.

(2) $[a]^*$ is an S-ideal, for all $a \in L$.

(3) $PSI(L)$ is 0-distributive semilattice.

Recall that, every pseudo-complemented ASL is 0-distributive ASL, but, converse is not true (see dia-1). Also, in a 0-distributive ASL not every $[x]^*$ is a principal S-ideal. For, in an ASL, dia - 1, $[a]^*$ is not a principal S-ideal. However, if $[a]^*$ is a principal S-ideal in a 0-distributive ASL, then we prove the following.

Theorem 3.10 : Let L be 0-distributive ASL. Then L is pseudo-complemented ASL if and only if $[a]^*$ is a principal S-ideal, for all $a \in L$.

Proof : Suppose L is pseudo-complemented ASL. Let $a \in L$. Then we have $t \in [a]^* \Leftrightarrow t\,o\,a = 0 \Leftrightarrow a\,o\,t = 0 \Leftrightarrow a^*\,o\,t = t \Leftrightarrow t \in (a]^*$. Thus $[a]^*$ is a principal S-ideal. Conversely, suppose $[a]^*$ is a principal S-ideal, for all $a \in L$. Now, we shall prove that L is a pseudo-complemented ASL. Let $a \in L$. Then by assumption $[a]^*$ is a principal S-ideal. Hence there exists $x \in L$ such that $[a]^* = (x]$. Now, put $a^* = x$. Then clearly, $*$ is a pseudo-complementation on L.

It can be easily seen that, for every nonempty subset A of an ASL L with 0, $A^* = \bigcap_{a \in A} [a]^*$. Now, we prove the following.

Theorem 3.11 : Let L be an ASL with 0, in which intersection of any family of S-ideals is again an S-ideal. Then the following are equivalent :

(1) L is 0-distributive ASL.

(2) A^* is an S-ideal, for all $A(\neq \phi) \subseteq L$.

(3) $SI(L)$ pseudo- complemented semilattice.

(4) $SI(L)$ is 0-distributive semilattice.

(5) $PSI(L)$ is 0-distributive semilattice.

Proof : $(1) \Rightarrow (2)$: Suppose L is 0-distributive ASL and suppose $A(\neq \phi) \subseteq L$. Now, we shall prove that A^* is an S-ideal. Since $a\,o\,0 = 0$ $for\,all\,a \in A, 0 \in A^*$. Hence A^* is nonempty. Let $x \in A^*$ $and\,t \in L$. Then $x\,o\,a = 0$, $for\,all\,a \in A$. Now, let $b \in A$. Then $(x\,o\,t)\,o\,b = (t\,o\,x)\,o\,b = t\,o\,(x\,o\,b) = t\,o\,0 = 0$. Therefore $x\,o\,t \in A^*$. Let $x, y \in A^*$. Then $x\,o\,a = 0, y\,o\,a = 0$, $for\,all\,a \in A$. Therefore $x, y \in [a]^*$, $for\,all\,a \in A$. It follows that $x, y \in \bigcap_{a \in A} [a]^*$, which is an S-ideal by hypothesis. Hence there exists $c \in A^*$ such that $c\,o\,x = x\,and\,c\,o\,y = y$. Thus A^* is an S-ideal.

$(2) \Rightarrow (3)$: Assume (2). Now, we shall prove that $SI(L)$ is pseudo-complemented semilattice. Let $I \in SI(L)$. Then we have I^* is an S-ideal. Clearly, $I \cap I^* = \{0\}$. Let $J \in SI(L)$ such that $I \cap J = (0]$. Now, we shall prove that $J \subseteq I^*$. Let $x \in J\,and\,a \in I$. Then we have $x\,o\,a \in J$ and also $x\,o\,a \in I$. Therefore $x\,o\,a \in I \cap J = (0]$. It follows that $x \in I^*$. Hence $J \subseteq I^*$. Thus $SI(L)$ is pseudo-complemented semilattice. $(3) \Rightarrow (4)$ is clear.

$(4) \Rightarrow (5)$: Assume (4). Now, we shall prove that $PSI(L)$ is 0-distributive semilattice. Let $(a], (b], (c] \in PSI(L)$ such that $(a] \cap (b] = (0], (a] \cap (c] = (0]$. Since $SI(L)$ is 0-distributive semilattice, there exists $I \in SI(L)$ such that $I \supseteq (b], (c]\,and\,I \cap (a] = (0]$. Therefore $I \cap (b] = (b], I \cap (c] = (c]\,and\,I \cap (a] = (0]$. This implies that $b, c \in I$. Since I is an S-ideal, there exists $d \in I$ such that $d\,o\,b = b$, $d\,o\,c = c$. It follows that $(d] \cap (b] = (d\,o\,b] = (b]$, $(d] \cap (c] = (d\,o\,c] = (c]$. Hence $(d] \supseteq (b], (c]$. Now, we have $d\,o\,a \in I$ and $d\,o\,a \in (a]$. Therefore $d\,o\,a \in I \cap (a] = (0]$. Hence $d\,o\,a = 0$. This implies that $(d] \cap (a] = (d\,o\,a] = (0]$. Thus $PSI(L)$ is 0-distributive semilattice. $(5) \Rightarrow (1)$ follows by theorem 3.7.

In the following we derive necessary and sufficient condition for a proper filter to become a maximal filter.

Theorem 3.12 : Let L be an ASL with 0. A proper filter M of L is maximal if and only if for any $a \in L - M$, there exists $b \in M$ such that $a\,o\,b = 0$.

Proof : Suppose M is a maximal filter in L. Let $a \in L - M$. Now, put $M' = \{y \in L : y\,o\,(a\,o\,b) = a\,o\,b$ $for\,some\,b \in M\}$. Now, we shall prove that M' is a filter. Clearly, $M' \neq \phi$ since $M \neq \phi$. Let $t_1, t_2 \in M'$. Then $t_1\,o\,(a\,o\,b_1) = a\,o\,b_1\,and\,t_2\,o\,(a\,o\,b_2) = a\,o\,b_2$ $for\,some\,b_1, b_2 \in M$ and hence $b_1\,o\,b_2 \in M$.

Consider, $(t_1\,o\,t_2)\,o\,(a\,o\,(b_1\,o\,b_2)) = (t_1\,o\,t_2)\,o\,((a\,o\,a)\,o\,(b_1\,o\,b_2))$

$= (t_1\,o\,t_2)\,o\,((a\,o\,(a\,o\,(b_1\,o\,b_2)))$

$$=(t_1 o t_2) o (a o((a o b_1) o b_2))$$
$$= (t_1 o t_2) o(a o((b_1 o a) o b_2))$$
$$=(t_1 o t_2) o (a o(b_1 o(a o b_2)))$$
$$=(t_1 o t_2) o((a o b_1) o (a o b_2))$$
$$= t_1 o(t_2 o ((a o b_1) o (a o b_2)))$$
$$= t_1 o ((t_2 o (a o b_1)) o (a o b_2))$$
$$= t_1 o (((a o b_1) o t_2) o (a o b_2))$$
$$=(t_1 o ((a o b_1) o t_2)) o (a o b_2)$$
$$=(t_1 o (a o b_1)) o t_2) o (a o b_2)$$
$$=(t_1 o (a o b_1)) o(t_2 o (a o b_2))$$
$$=(a o b_1) o (a o b_2)$$
$$=(a o b_1) o a) o b_2$$
$$=(a o(a o b_1)) o b_2$$
$$= ((a o a) o b_1) o b_2$$
$$= (a o b_1) o b_2$$
$$= a o(b_1 o b_2)$$

Therefore $t_1 o t_2 \in M'$. Now, let $t \in M'$ and $x \in L$ such that $x o t = t$. Then $t o (a o b) = a o b$ for some $b \in M$. Now, consider $x o (a o b) = x o (t o (a o b)) = (x o t) o (a o b) = t o (a o b) = a o b$. Therefore $x \in M'$. Hence M' is a filter. Clearly, $M \subseteq M'$. Now, for any $b \in M$, we have $a o (a o b) = a o b$. Therefore $a \in M'$ and $a \notin M$. Hence $M \subsetneq M'$. Since M is maximal filter, $M' = L$. Again, since $0 \in L = M'$, $0 \in M'$. It follows that $0 o (a o b) = a o b$ for some $b \in M$. Therefore $a o b = 0, b \in M$. Conversely, assume the condition. Now, we shall prove that M is a maximal filter. Suppose H is a filter of L such that $M \subsetneq H$. Then there exists $a \in H$ such that $a \notin M$. Therefore by assumption there exists $b \in M$ such that $a o b = 0$. It follows that $0 = a o b \in M$. Thus $H = L$. Therefore M is maximal.

First, observe that in an ASL L, maximal filter need not be a prime filter. For, consider the following example.

Example 3.13 : Let $L = \{0, a, b, c, d\}$ and define a binary operation o on L as follows:

o	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
d	0	a	b	c	d

Then clearly, L is an ASL. Consider, the filter $[a) = \{a, d\}$. Then clearly, $[a)$ is a maximal filter. Now, consider the filters $[b) = \{b, d\}$ and $[c) = \{c, d\}$. Then clearly, $[b) \cap [c) = \{d\} \subseteq \{a, d\} = [a)$. But, $[b) \nsubseteq [a)$ and $[c) \nsubseteq [a)$. Therefore $[a)$ is not a prime filter. However, we prove the following.

Theorem 3.14 : Let L be 0-distributive ASL. Then every maximal filter of L is a prime filter.

Proof : Suppose F is a maximal filter. Now, we shall prove that F is a prime filter. Let F_1, F_2 be filters of L. Suppose $F_1 \nsubseteq F$ and $F_2 \nsubseteq F$. Then there exists $x \in F_1$ such that $x \notin F$ and $y \in F_2$ such that $y \notin F$. Therefore by theorem 3.12, there exists $a, b \in F$ such that $a o x = 0, b o y = 0$. It follows that $(a o b) o x = 0, (a o b) o y = 0$.

Since L is 0-distributive, there exists $d \in L$ such that $d o x=x, d o y=y$ and $d o(a o b)=0$. Now, we have $d o x=x$ and $x \in F_1$. Hence $d \in F_1$. Similarly, $d \in F_2$. Therefore $d \in F_1 \cap F_2$. Since $a, b \in F$, $a o b \in F$. Now, if $d \in F$ then $d o(a o b) \in F$. It follows that $0 \in F$. Hence $F = L$, a contradiction. Therefore $d \notin F$. Hence $F_1 \cap F_2 \nsubseteq F$. Thus F is a prime filter.

In the following, we give some properties of 0-distributive ASL.

Theorem 3.15 : Let L be a 0-distributive ASL, in which intersection of any family of S-ideals is again an S-ideal. Then we have the following:

1. For any S-ideal I in L, the set I^{**} is the largest among all the S-ideals J in L with property that for every $x(\neq 0) \in J$, there exists an element $y\,(\neq 0) \in I$ such that $y \leq x$.
2. Let $I_1, I_2, \ldots, I_n$ be S-ideals of an ASL L. Then $(\bigcap_{i=1}^{n} I_i)^{**} = \bigcap_{i=1}^{n} I_i^{**}$.

Proof :

1. Suppose I is an S-ideal of L. Then clearly, I^{**} is an S-ideal of L. Let $x(\neq 0) \in I^{**}$. Suppose there exists no nonzero element $y \in I$ such that $y \leq x$. Now, we shall prove that $x o z=0$, $for\, all\, z \in I$. Suppose $x o z \neq 0$ for some $z \in I$. Then we have $z o x \neq 0$, $z o x \in I\, and\, z o x \leq x$, a contradiction to our assumption. Therefore $x o z=0$, for all $z \in I$. It follows that $x \in I^{*}$. Since $x \in I^{**}$, we get $x \in I^{*} \cap I^{**} = \{0\}$, a contradiction to $x \neq 0$. Thus there exists $y(\neq 0) \in I$ such that $y \leq x$. Suppose J is an S-ideal of L with the property that for every $x \neq 0 \in J$, there exists $y \neq 0 \in I$ such that $y \leq x$. Now, we shall prove that $J \subseteq I^{**}$. Suppose $J \nsubseteq I^{**}$. Then there exists $x \in J$ such that $x \notin I^{**}$. Thus $x o y \neq 0$ for some $y \in I^{*}$. Since $x \in J, x o y\,(\neq 0) \in J$. Hence there exists $z\,(\neq 0) \in I$ such that $z \leq x o y$. Since $y \in I^{*}, x o y \in I^{*}$. It follows that $z \in I^{*}$. Also, we have $z \in I$ and hence $z \in I^{*} \cap I^{**} = \{0\}$. Therefore $z = 0$, a contradiction. Thus $J \subseteq I^{**}$.
2. We prove this result by using induction on n. If $n=1$, then the result is clear. Now, we shall prove this result for $n=2$. Let $x(\neq 0) \in I_1^{**} \cap I_2^{**}$. Then $x \in I_1^{**}$ and $x \in I_2^{**}$. Since $x(\neq 0) \in I_1^{**}$, by (1), there exists $y(\neq 0) \in I_1$ such that $y \leq x$. It follows that $y \in I_1^{**} \cap I_2^{**}$. Again, since $y(\neq 0) \in I_2^{**}$, by (1), there exists $z\,(\neq 0) \in I_2$ such that $z \leq y$. Therefore $z \in I_1$ and hence $z \in I_1 \cap I_2$, $z \leq x$. It follows by condition (1), we get $I_1^{**} \cap I_2^{**} \subseteq (I_1 \cap I_2)^{**}$. Conversely, we have $I_1 \cap I_2 \subseteq I_1, I_2$. It follows that $(I_1 \cap I_2)^{**} \subseteq I_1^{**} \cap I_2^{**}$. Thus $(I_1 \cap I_2)^{**} = I_1^{**} \cap I_2^{**}$. Now, assume that the result is true for $n-1$. That is $(\bigcap_{i=1}^{n-1} I_i)^{**} = \bigcap_{i=1}^{n-1} I_i^{**}$. Consider, $(\bigcap_{i=1}^{n} I_i)^{**} = (\bigcap_{i=1}^{n-1} I_i \cap I_n)^{**} = (\bigcap_{i=1}^{n-1} I_i)^{**} \cap I_n^{**} =$. $\bigcap_{i=1}^{n-1} I_i^{**} \cap I_n^{**} = \bigcap_{i=1}^{n} I_i^{**}$.

Recall that an element a in an ASL L is said to be dense if $[a]^{*} = \{0\}$. Also, it can be easily seen that every unimaximal element in an ASL L is a dense element. In the following, we prove some more properties of 0-distributive ASLs. First, we need the following.

Lemma 3.16 : The set D, of all dense elements in an ASL L with unimaximal element is a filter.

Proof : Suppose L has unimaximal element say m. Then we have m is a dense element and hence $m \in D$. Therefore $D \neq \phi$. Now, let $a, b \in D$. Then $[a]^{*} = \{0\}, [b]^{*} = \{0\}$. Now, we shall prove that $[a o b]^{*} = \{0\}$. Let $t \in [a o b]^{*}$. Then $t o a o b = 0$. Therefore $t o a \in [b]^{*} = \{0\}$. It follows that $t \in [a]^{*} = \{0\}$. Hence $t = 0$. Therefore $[a o b]^{*} = \{0\}$. Thus $a o b \in D$. Let $a \in D\, and\, t \in L$ such that $t o a = a$. Since $a \in D$, $[a]^{*} = \{0\}$. Hence $[t o a]^{*} = \{0\}$. Now, we shall prove that $[t]^{*} = \{0\}$. Let $x \in [t]^{*}$. Then $x o t = 0$. Therefore $(x o t) o a = 0$. This implies that $x o (t o a) = 0$. Hence $x \in [t o a]^{*} = \{0\}$. It follows that $x = 0$. Therefore $[t]^{*} = \{0\}$. Hence $t \in D$. Thus D is a filter.

Lemma 3.17 : Let L be a 0-distributive ASL and $a,b\in L$ such that $a\circ d=b\circ d$ for some $d\in D$. Then $[a]^*=[b]^*$.

Proof: Suppose L is a 0-distributive ASL and $a,b\in L$ such that $a\circ d=b\circ d$ for some $d\in D$. Then $[a]^{**}=[a]^{**}\cap L=[a]^{**}\cap[d]^{**}(\text{since}[d]^*=\{0\},[d]^{**}=[0]^*=L)=[a\circ d]^{**}=[b\circ d]^{**}=[b]^{**}\cap [d]^{**}=[b]^{**}\cap L=[b]^{**}$. It follows that $[a]^{***}=[b]^{***}$. Thus $[a]^*=[b]^*$.

Theorem 3.18 : In a 0-distributive ASL L if $A\neq\{0\}$ is the intersection of all non-zero S-ideals of L, then $A^*=L-D$ *i.e.*, $A^*=\{x\in L:[x]^*\neq\{0\}\}$.

Proof : Put $T=\{x\in L:[x]^*\neq\{0\}\}$. Now, we shall prove that $A^*=T$. Let $t\in A^*$. Then $t\circ a=0$, for all $a\in A$. Since $A\neq\{0\}$, choose $a_0(\neq 0)\in A$. Then $t\circ a_0=0$. Therefore $a_0\in[t]^*$. Hence $[t]^*\neq\{0\}$. Thus $t\in T$. Conversely, suppose $t\in T$. Then $[t]^*\neq\{0\}$. Therefore $A\subseteq[t]^*$. It follows that $[t]^{**}\subseteq A^*$. Now, we have $t\in[t]^{**}\subseteq A^*$ and hence $t\in A^*$. Therefore $T\subseteq A^*$. Thus $T=A^*$.

Next, we prove that if F is a filter and I is an annihilator ideal of a 0-distributive ASL L, which are disjoint, then there exists a prime filter H of L containing F and disjoint with I. That is $F\subseteq H$ and $H\cap I=\phi$.

Theorem 3.19 : Let L be a 0-distributive ASL with unimaximal element in which intersection of any family of S-ideals is again an S-ideal. Then for any filter F of L and for any annihilator ideal I of L such that $F\cap I=\phi$, there exists a prime filter containing F and disjoint with I.

Proof : Suppose L is a 0-distributive ASL. Let F be a filter of L and I be an annihilator ideal of L such that $F\cap I=\phi$. Now, put $\mathfrak{F}=\{H : H \text{ is a filter of } L, F\subseteq H \text{ and } H\cap I\neq\emptyset\}$. Then $\mathfrak{F}\neq\emptyset$, since $F\in\mathfrak{F}$. Clearly $\mathfrak{F}$ is a poset under set inclusion and it can be easily verified that $\mathfrak{F}$ satisfies the hypothesis of Zorn's lemma. Therefore, by Zorn's lemma, $\mathfrak{F}$ has maximal element say Q. Then clearly, Q is a proper filter, $F\subseteq Q$ and $Q\cap I=\phi$. First, we shall prove that $I^*\subseteq Q$. Let $x\in I^*$. Then $x\circ i=0$, for all $i\in I$. Suppose $x\notin Q$. Now, suppose $(Q\vee[x))\cap I\neq\phi$. Then there exists $t\in L$ such that $t\in Q\vee[x)$ and $t\in I$. Therefore $t\circ(a\circ b)=a\circ b$, for some $a\in Q, b\in[x)$ and $t\in I$. Hence $(t\circ(a\circ b))\circ x=(a\circ b)\circ x$. This implies $((a\circ b)\circ t)\circ x=(a\circ b)\circ x$. Therefore $(a\circ b)\circ(t\circ x)=a\circ(b\circ x)$. It follows that $0=a\circ x$. Thus $a\in I^{**}=I$. But, we have $a\in Q$. Therefore $Q\cap I\neq\phi$, a contradiction to $Q\cap I=\phi$. Hence $(Q\vee[x))\cap I=\phi$. This implies $Q\vee[x)\in\mathfrak{F}$ and $Q\subsetneq Q\vee[x)$, a contradiction to Q is maximal in $\mathfrak{F}$. Hence $I^*\subseteq Q$. Let $z\in L$ such that $z\notin Q$. Then $(Q\vee[z))\cap I\neq\phi$. Therefore there exists $t\in L$ such that $t\in Q\vee[z)$ and $t\in I$. This implies $t\circ(a\circ b)=a\circ b$, for some $a\in Q, b\in[z)$ and $t\in I$. It follows that $t\circ(a\circ b)=a\circ b, a\in Q, b\circ z=z$ and $t\in I$. Now, let $s\in I^*$. Then $s\in Q$ and hence $a\circ s\in Q$. Now, we have $t\circ(a\circ b)=a\circ b$. This implies $(t\circ(a\circ b))\circ s=(a\circ b)\circ s$. It follows that $(a\circ b)\circ s=0$. Hence $((a\circ b)\circ s)\circ z=0$. It follows that $(a\circ s)\circ z=0$. Therefore $(a\circ s)\circ z=0$, $a\circ s\in Q$. Hence by theorem 3.12, Q is a maximal filter. Since L is 0-distributive, by theorem 3.14, Q is a prime filter.

Corollary 3.20 : Any two non-zero elements a, b for which $a\circ b=0$ are separated by a prime filter in a 0-distributive ASL.

Proof : Suppose $a\circ b=0$. Then $a\in[b]^*$. Suppose $[b)\cap[b]^*\neq\phi$. Then we can choose $t\in[b)$ and $t\in[b]^*$. Therefore $b=t\circ b$ and $t\circ b=0$. It follows that $b=0$, a contradiction. Hence $[b)\cap[b]^*=\phi$. Therefore by theorem 3.19, there exists a prime filter, say F such that $[b)\subseteq F$ and $[b]^*\cap F=\phi$. It follows that $b\in F$ and $a\notin F$.

Corollary 3.21 : In a 0-distributive ASL L, any nonzero element of L is separated from the set of its disjoint elements by a prime filter.

Proof : Suppose $a(\neq 0)\in L$. Then we have $[a]^*=\{x\in L: a\circ x=0\}$. Now, put $F=[a)$. Suppose $t\in[a)\cap[a]^*$ Then $t\in[a)$ and $t\in[a]^*$. It follows that $a=0$, a contradiction to $a\neq 0$. Therefore $[a)\cap[a]^*=\phi$. Hence by theorem 3.19, there exists a prime filter say H of L such that $[a)\subseteq H$ and $[a]^*\cap H=\phi$. Therefore $a\in H$ and $[a]^*\cap H=\phi$.

It can be easily seen that if L is an ASL with 0, then the set $\{[a]^{**} : a \in L\}$ is a semilattice with respect to set intersection.

Definition 3.22: Let L be a 0-distributive ASL and define a map $f : L \to \{[a]^{**} : a \in L\}$ by $f(a) = [a]^{**}$.

It can be easily verified that f is an ASL homomorphism. Now, we introduce the concept of disjunctive ASL and derive a necessary and sufficient condition for a 0-distributive ASL to become a disjunctive ASL. For, this we need the following.

Lemma 3.23 : Let L be 0-distributive ASL. Then $f([a]^*) = \{[b]^{**} : a \circ b = 0\}$.

Proof : Put $H = \{[b]^{**} : a \circ b = 0\}$. Now, we shall prove that $f([a]^*) = H$. Suppose $f(t) \in f([a]^*)$. Then $t \in [a]^*$. Therefore $t \circ a = 0$. Hence $[t]^{**} \in H$. Thus $f(t) \in H$. Conversely, suppose $[b]^{**} \in H$. Then $a \circ b = 0$. Therefore $b \in [a]^*$. Hence $f(b) \in f([a]^*)$. Thus $[b]^{**} \in f([a]^*)$.

Lemma 3.24 : Let L be 0-distributive ASL. Then $\{[b]^{**} : [a]^{**} \cap [b]^{**} = \{0\}\} = [f(a)]^*$.

Proof : Put $P = \{[b]^{**} : [a]^{**} \cap [b]^{**} = \{0\}\}$. Now, we shall prove that $P = [f(a)]^*$. Suppose $[b]^{**} \in P$. Then $[a]^{**} \cap [b]^{**} = \{0\}$. This implies $f(a) \cap [b]^{**} = \{0\}$. Hence $[b]^{**} \in [f(a)]^*$. Therefore $P \subseteq [f(a)]^*$. Conversely, suppose $[b]^{**} \in [f(a)]^*$. Then $[b]^{**} \cap f(a) = \{0\}$. It follows that $[b]^{**} \cap [a]^{**} = \{0\}$. Thus $[a]^{**} \in P$. Therefore $[f(a)]^* \subseteq P$. Thus $[f(a)]^* = P$.

Theorem 3.25 : Let L be 0-distributive ASL. Then $f(a) = \{0\}$ if and only if $a = 0$. Moreover $f([a]^*) = [f(a)]^*$.

Proof : Suppose $f(a) = \{0\}$. Then $[a]^{**} = \{0\}$. Therefore $[a]^{***} = \{0\}^*$. It follows that $[a]^* = L$ and hence $a = 0$. Conversely, suppose $a = 0$. Then $f(a) = f(0) = [0]^{**} = \{0\}$. Hence $f(a) = \{0\}$. Since f is an ASL homomorphism, we get $a \circ b = 0$ if and only if $f(a \circ b) = f(a) \circ f(b) = \{0\}$. Thus $f([a]^*) = \{[b]^{**} : a \circ b = 0\} = \{[b]^{**} : [a]^{**} \cap [b]^{**} = \{0\}\} = [f(a)]^*$.

In general, in an ASL with 0, if $[a]^* = [b]^*$ need not imply $a = b$. For, consider the following example.

Example 3.26 : Let $L = \{0, a, b, c, d\}$ and define a binary operation o on L as follows:

o	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	a	a
b	0	0	b	0	b
c	0	a	0	c	c
d	0	a	b	c	d

Then clearly (L, o) is an ASL with 0. In this ASL, clearly $[a]^* = [c]^*$, but $a \neq c$. Now, we define the following.

Definition 3.27 : An ASL L with 0 is said to be disjunctive if for any $a, b \in L$, $[a]^* = [b]^*$, then $a = b$.

Example 3.28 : Let $L = \{0, a, b, c, d\}$ and define a binary operation o on L as follows:

o	0	a	b	c	d
0	0	0	0	0	0
a	0	a	0	0	a
b	0	0	b	0	b
c	0	0	0	c	c
d	0	a	b	c	d

Then clearly (L,o) is an ASL with 0 and clearly is a disjunctive ASL.

Now, we give necessary and sufficient condition for a 0-distributive ASL to become a disjunctive ASL.

Theorem 3.29 : Let L be 0-distributive ASL. Then the following are equivalent:

1. f is injective.
2. L is disjunctive.

Proof : (1)$\Rightarrow$(2): Assume (1). Suppose $a,b\in L$ such that $[a]^{*}=[b]^{*}$. Then $[a]^{**}=[b]^{**}$. Therefore $f(a)=f(b)$. Hence $a=b$. since f is injective. Thus L is disjunctive.

(2)$\Rightarrow$(1): Assume (2). Let $a,b\in L$ such that $f(a)=f(b)$. Then $[a]^{**}=[b]^{**}$. This implies $[a]^{***}=[b]^{***}$ and hence $[a]^{*}=[b]^{*}$. Thus $a=b$, since L is disjunctive. Therefore f is injective.

It can be easily seen that, in the example 3.26, we have $[a]^{*}=[c]^{*}$, but $(a]\neq(c]$. Now, we introduce the concept of weakly disjunctive ASL. We obtain a necessary and sufficient condition for a 0-distributive ASL to become weakly disjunctive ASL. First, we need the following.

Definition 3.30 : An ASL L with 0 is said to be weakly disjunctive ASL if for any $a,b\in L$, $[a]^{*}=[b]^{*}$, then $(a]=(b]$.

It can be easily observe that every disjunctive ASL is weakly disjunctive ASL. But, the converse is not true. For, consider the following example.

Example 3.31 : Let $L=\{0,a,b,c\}$ be a discrete ASL. Then we have every non-zero element in L is unimaximal. But, we have every unimaximal element is dense. Hence every non-zero element in L is dense. Also, we have, if x is unimaximal element in L, then $(x]=L$. It follows that L is weakly disjunctive ASL. But, L is not disjunctive ASL, since $[a]^{*}=\{0\}=[b]^{*}$, but $a\neq b$.

Finally, we obtain a necessary and sufficient condition for a 0-distributive ASL to become weakly disjunctive ASL. First, we need the following.

Definition 3.32 : Let L be 0-distributive ASL and define map $h:L\to PSI(L)$ by $h(a)=(a]$ for all $a\in L$.

It can be easily seen that h is an ASL homomorphism. Now, we prove the following.

Lemma 3.33 : Let L be a 0-distributive ASL. Then $h(a)=0$ if and only if $a=0$. Moreover, $h([a]^{*})=[h(a)]^{*}$.

Proof : Suppose $h(a)=0$. Then $(a]=0$. Therefore $a=0$. Conversely, suppose $a=0$. Then $(a]=0$. It follows that $h(a)=0$. Let $h(t)\in h([a]^{*})$. Then $t\in[a]^{*}$. It follows that $toa=0$. Hence $h(t)oh(a)$ $=h(toa)=h(0)=(0]$. Therefore $h(t)=[h(a)]^{*}$. Thus $h([a]^{*})\subseteq[h(a)]^{*}$. Conversely, suppose $(t]\in[h(a)]^{*}$. Then $(t]\cap h(a)=(0]$. Now, we have $h(toa)=h(t)\cap h(a)=(0]$. This implies that $h(toa)=(0]$. Hence $(toa]=(0]$. It follows that $toa=0$. Hence $t\in[a]^{*}$. This implies $h(t)\in h([a]^{*})$. It follows that $(t]\in h([a]^{*})$. Therefore $[h(a)]^{*}\subseteq h([a]^{*})$. Thus $[h(a)]^{*}=h([a]^{*})$.

Lemma 3.34 : Let L be 0-distributive ASL and define $g:PSI(L)\to\{[a]^{**}:a\in L\}$ by $g((a])=[a]^{**}$, for all $(a]\in PSI(L)$. Then g is an ASL homomorphism.

Proof : Let $(a],(b]\in PSI(L)$ such that $(a]=(b]$. Now, we shall prove that $g((a])=g((b])$. It is enough to prove that $(a]^{*}=[a]^{*}$. Let $t\in(a]^{*}$. Then $tos=0$, for all $s\in(a]$. In particular $toa=0$, since $a\in(a]$. Hence $t\in[a]^{*}$. Therefore $(a]^{*}\subseteq[a]^{*}$. Conversely, suppose $t\in[a]^{*}$. Then $toa=0$. Let $s\in(a]$. Then $s=aos$. Now, consider $tos=to(aos)=(toa)os=0os=0$. It follows that $t\in(a]^{*}$. Therefore $[a]^{*}\subseteq(a]^{*}$. Hence $(a]^{*}=[a]^{*}$. Thus g is well-defined. Let $(a],(b]\in PSI(L)$. Consider, $g((a]\cap(b])=g((aob])=[aob]^{**}=[a]^{**}\cap[b]^{**}=g((a])\cap g((b])$ and $g((0])=[0]^{**}=\{0\}$. Therefore g is an ASL homomorphism.

Finally, we prove the following.

Theorem 3. 35 : Let L be 0-distributive ASL. Then the following are equivalent:

(1) g is injective.

(2) L is weakly disjunctive ASL.

Proof : $(1)\Rightarrow(2)$: Assume (1). Let $a,b\in L\ such that [a]^{*}=[b]^{*}$. Then $[a]^{**}=[b]^{**}$. Therefore $g((a])=g((b])$. Hence $(a]=(b]$. Thus L is weakly disjunctive ASL.

$(2)\Rightarrow(1)$: Assume (2). Let $(a],(b]\in PSI(L)$ such that $g((a])=g((b])$. Then $[a]^{**}=[b]^{**}$. It follows that $[a]^{***}=[b]^{***}$. This implies that $[a]^{*}=[b]^{*}$. Therefore $(a]=(b]$. Thus g is injective.

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