CONSTRUCTION OF DEFICIENT DISCRETE QUARTIC SPLINE

ROHIT DUBEY
MATHEMATICS DEPARTMENT
GLOBAL ENGGINERING COLLEGE JABALPUR M.P. INDIA
Y.P.DUBEY
MATHEMATICS DEPARTMENT
GOVT SCIENCE AND ART COLLEGE KUNDAM JABALPUR M.P. INDIA.

ABSTRACT. We have constructed deficient discrete quartic spline interpolation and obtained error bounds with function .

I INTRODUCTION.

There are two basic method to developing splines - the *vibrational method*
Where in splines are defined as the solutions of certain constrained minimization problems,[see 4,5] and the *constructive* method where in they are defined by piecing together classes of functions at certain 'knots'. In the very first paper on discrete splines [14], the vibrational method has been used and discrete splines are introduced as solutions to constrained minimization problems in real Euclidean space.The constructive method has been employed in the work of [2,3,4,]. Both
Schumaker [12] and Lyche [2,3] deal with discrete polynomial splines. Rana and Dubey [11,12,13] have obtained local behavior of discrete cubic spline and best error bounds of Quantic and quartic splines respectively. Noor and Khalifa [8,9,10] have found Cubic splines collocation methods for unilateral problems .Finite difference technique for solving obstacle Problems given by Noor and Tirmizi and for different aspect see [1,6, 7]. In this paper we shall develop a discrete *quartic spline* via a constructive method and obtained error between function and its discrete quartic spline, also error of first differences.

II EXISTENCE AND UNIQUENESS.

Let us consider a mesh on $[0,1]$ which is defined by

$$0 = x_0 < x_1 < ... < x_n = 1$$

with $x_i - x_{i-1} = P,$ for $i = 1,2,...,n.$ and $P \geq h,$ throughout $h,$ will be represent a given the real number. Let s_i be the restriction in $[x_{i-1}, x_i]$ of $s(x,h)$ over interval $[0,1]$ is a polynomial of degree 4 or less for $i = 1,2,...,n,$ then $s(x,h)$ defines a discrete quartic spline if

$$D_h^{\{j\}} s_i(x_i, h) = D_h^{\{j\}} s_{i+1}(x_i, h) \quad j = 0,1,2. \qquad (2.1)$$

where the forword difference operator D_h are defined as

$$D_h^{\{0\}} f(x) = f(x),$$

$$D_h^{\{1\}} f(x) = \frac{f(x+h) - f(x-h)}{2h}$$

$$D_h^{\{2\}} f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}.$$

Let $S(4,1,\Delta,h)$ is the class of deficient discrete quartic splines where in $S^*(4,1,\Delta,h)$ denotes the class of all discrete deficient quatic splines which satisfies the boundary conditions

$$D_h^{\{1\}} s(x_0,h) = D_h^{\{1\}} f(x_0,h), \qquad D_h^{\{1\}} s(x_n,h) = D_h^{\{1\}} f(x_n,h), \qquad (2.2)$$

PROBLEM 1. Given $h > 0$, for what restriction on P does there exists a unique $s(x,h) \in S^*(4,1,P,h)$ which satisfies the conditions.

$$s(x_i,h) = f(x_i), \qquad i = 0,1,...,n. \qquad (2.3)$$

$$s(\alpha_i,h) = f(\alpha_i), \qquad i = 1,...,n.$$

where $\alpha_i = x_{i-1} + (3/4)P$

Let $A(z)$ be a quartic polynomial defined on $[0,1]$, it can be easily verified that

$$A(Z) = A(0)\ q_1(Z) + A(3/4)\ q_2(Z) + A(1)\ q_3(Z) + D_h^{\{1\}} A(0)\ q_4(Z) + D_h^{\{1\}} A(1)\ q_5(Z) \qquad (2.4)$$

where $q_1(Z) = [1 - (67/9)z^2 + (98//9)z^3 - (40/9)z^4]$

$q_2(Z) = 256z^2(1-z)^2/9$

$q_3(z) = z^2 (3-4z) (-16-3(3+4z)- (3/2)z)$

$q_4(Z) = [z\ \ (10/3)z^2 + (11/3)z^3 - 4z^4/3]$

$q_5(Z) = z^2(3-4z)(1-z)$

Proof of problem 1 includes in the following theorem

THEOREM 2.1. There exists a unique deficient discrete quartic spline interpolant s in $A^*(4,1,P,h)$ satisfies the conditions (2.2) - (2.3) and boundary condition (2.1)

Let $x = x_i + p\ t, \quad 0 \le t \le 1$, then view of conditions (2.1) – (2.3). We can expressed (4.2.4) in terms of the restriction s_i of s to $[x_i, x_{i+1}]$ as follows.

$$s_i(x) = f(x_i)A_1(t) + f(\alpha_{i+1})A_2(t) + f(x_{i+1})A_3(t) + D_n^{\{1\}} f(x_i)A_4(t) + D_n^{\{1\}} f(x_{i+1})A_5(t)$$

(2.5)

Let $H(a,b) = ap_i^2 + bh^2$, a,b are real numbers.

Since $s \in C^2[0,1]$ and setting $D_h^{\{1\}} s(x_i) = m_i$, we have from (2.5)

$$-\frac{4}{3} H(3/4,1)\ m_{i-1} + \{4H(3/2,1) + 4H(5/2,1)/3\} m_i - \frac{4}{1} H(3/4,1) m_{i+1} = F_i \qquad \text{for } i = 0,1,...,n-1.$$

where $$F_i = f_i\{16H(-219/16,\ 3/2) + \frac{16}{9} H(67/16,\ 5/2)\}$$

$$-\frac{16 f(\alpha_i)}{3} H(1,1) + \frac{16 f(\alpha_{i+1})}{3} H(1,1) - \frac{16}{3} f_{i-1} H(67/16,\ 5/2) +$$

$$+16 f_{i+1} H(21/16,\ 3/2)] \qquad (2.6)$$

Clearly in (2.4) the coefficients of m_i is positive and coefficients of $m_{i-1}^{\}}$ and m_{i+1} are negative. Therefore the excess of absolute values of coefficients of $m_i^{\}}$ over the sum of absolute values of coefficients of $m_{i-1}^{\}}$ and $m_{i+1}^{\}}$ is given by

$$\frac{16}{3}[\{(17/16)p_i^2 + \{(1/16)p^2,1) + h^2\}]$$

which is positive .
Thus the coefficient matrix of the system of equations (2.6) is diagonally dominant and hence invertible, therefore the system of equation (2.6) has unique solution, which completes the proof of the theorem 2.1.

REMARK: The studies concerning discrete splines smaller values of h, have special significance for simple region that discrete spline reduce to continuous spline as $h \to 0$ and diffrence reduce into derivative

III ERROR BOUNDS.

For a given $h > 0$, we introduced the set

$$R_h = \{x_0 + jh : j \ is\ an\ integer\}$$

and define a discrete interval as follows

$$[0,1]_h = [0,1] \cap R_h$$

for a function f and discrete points x_1, x_2, x_3 in the domains. The first and second divided difference are defined as

$$[x_1, x_2]_f = \frac{f(x_2) - f(x_1)}{(x_2 - x_1)}$$

and

$$[x_1, x_2, x_3]_f = \frac{[x_2, x_3]_f - [x_1, x_2]_f}{(x_3 - x_1)}$$

respectively similarly, we can define the higher order divided difference.

Now in this section, we shall obtain the precise estimate of the error bounds for deficient discrete quartic spline interpolation and function i.e. $e = s - f$ over the discrete interval $[0,1]_h$

We shall need the following Lemma due to Lyche [2,3].

LEMMA 3.1. Let $\{a_i\}_{i=1}^n$ and $\{b_j\}_{j=1}^m$ be given sequence of non-negative real numbers such that $\sum_{i=1}^n a_i = \sum_{j=1}^m b_j$ then for any real valued function f, defines on discrete interval $[0,1]_h$, we have

$$\left| \sum_{i=1}^n a_i [x_{j_0}, x_{j_1}, ..., x_{j_k}]_f - \sum_{j=1}^n b_j [y_{j_0}, y_{j_1}, ..., y_{j_k}]_f \right|$$
$$\leq w(D_n^{(K)} f, |1 - kh|) \sum a_j$$

for relevant values of j, k .

We can write system of equation (2.3) as Matrics form

$$B(h)M(h) = F \qquad (3.1)$$

where $B(h)$ is the coefficients matrix and $M(h) = m_i(h)$. However, already shown in the proof of theorem 2.1. $B(h)$ is invertible. Denoting the inverse of $B(h)$ by $B^{-1}(h)$, we note that row max norm $\| B^{-1}(h) \|$ satisfies the following inequality

$$\| B^{-1}(h) \| \leq y(h) \qquad (3.2)$$

where $y(h) = \max\{l_i(h)\}^{-1}$

For convenience, we write $f^{\{1\}}$ for $D_h^{\{1\}}f$, $f_i^{\{2\}}$ for $D_h^{\{2\}}f(x_i)$ and $w(f,p)$ for modules of continuity of f, the discrete norm of a function f over the interval $[0,1]_h$ is defined by

$$\| f \| = \max_{x\in[0,1]_h} |f(x)| \qquad (3.3)$$

We are now state the following:

THEOREM 3.1. Suppose $s(x,h)$ is the deficient discrete quartic splines interpolant of theorem 2.1 then

$$\| e(x) \| \le C_1(p,h) w(f^{\{1\}}, p) \qquad \textbf{(3.4)}$$

$$\| e^{\{1\}}(x) \| \le C_2(p,h) w(f^{\{1\}}, p) \qquad \textbf{(3.5)}$$

Where $C_1(p,h)$ and $C_2(p,h)$ are some positive functions of p and h.

PROOF OF THEOREM 3.1. Writing $f(x_i) = f_i$. Equation (3.1) may be written as

$$B(h)\cdot e^{\{1\}}(x_i) = F_i(h) - B(h) f_i^{\{1\}} = (L_i) \ (say) \qquad (3.7)$$

We shall estimate $L_i(f)$ by using Lemma 3.1, due to Lyche[3]. It may observe that the $L_i(f)$ is written as

$$|(L_i)| = \left| \sum_{i=1}^{4} a_i [x_{i0}, x_{i1}]_f - \sum_{j=1}^{4} b_j [y_{j0}, y_{j1}]_f \right. \qquad (3.8)$$

where

$$a_1 = \frac{256}{9} H((-189/256, -3/2)$$

$$a_2 = \frac{64}{9}[H(1,1)],$$

$$a_3 = 16H(7/16, 1/4),$$

$$a_4 = \frac{4}{3}[H(1/4, 1)]$$

$$b_1 = 3/4[\ H(5/2,\ 1)]$$

$$b_2 = \frac{256}{9}[9/16H(-219/16,\ 3/2)],$$

$$b_3 = (64/3)[H(1,1)],$$

$$b_4 = [H(6,\ 4)],$$

and

$$x_{10} = x_i = x_{20} = x_{21}, \quad x_{11} = x_{i+1},$$

$$x_{21} = \alpha_{i+1} = x_i + (3/4\ )p_i, \quad x_{30} = x_{i+1} - h,$$

$$x_{31} = x_{i+1} + h, \quad y_{20} = x_{i-1}, \quad y_{10} = x_i - h,$$

$$y_{11} = x_i + h, \quad y_{30} = x_{i-1}, \quad y_{31} = \alpha_i = x_{i-1} + (3/4)\ p_{i-1},$$

$$y_{40} = x_i - h, \quad y_{41} = x_i + h,$$

$$x_{40} = x_{i-1} - h, \quad x_{41} = x_{i-1} + h,$$

Since $a_1 + a_2 + a_3 = b_1$ and $a_4 = b_2 + b_3 + b_4$. Therefore

$$\sum_{i=1}^{4} a_i = \sum_{j=1}^{4} b_j = \frac{16}{3}[H(3/4\ ,1) + (1/4)H(1/4,1)]$$

Thus apply Lemma (3.1) for $n = 4, m = 4$ and $K = 1$

$$|L_i(f)| \le c^*(P,h)w(f^{\{1\}},P) \tag{3.10}$$

Now using the equations (3.2) and (3.10) in (3.7), we get

$$\| e^{\{1\}}(x\ ) \| \le y(h)c^*(P,h)w(f^{\{1\}},P) \tag{3.11}$$

This complete proof of inequality (3.6) of Theorem 3.1.

To obtain the bound of $e(x)$, we replace m_i by $e^{\{1\}}(x_i)$ in equation (2.5) to get

$$e(x) = p\ [e_i^{\{1\}} A_4(t) + e_{i+1}^{\{1\}} A_5(t)] + L_i^*(f) \tag{3.12}$$

where $L_i^*(f) = [f_i\ A_1(t) + f\ \ (\alpha_{i+1})A_2(t) + f_{i+1}A_3(t) + A_4(t)\, f_i^{\{1\}} + A_5(t)\, f_{i+1}^{\{1\}} - f(x)]$.

We write $L_i^*(f)$ in the form of divided difference as follows

$$|L_i^*(f)| = |\sum_{i=1}^{n} a_i - \sum_{j=1}^{m} b_j| \le \sum_{i=1}^{n} a_i = \sum_{j=1}^{m} b_j w(f^{\{1\}},P)$$

where

$$a_1 = (64/3)[(z^2 - 2z^3 - z^4)\ \ P\ ]$$

$$a_2 = P[-21z^2 + 46z^3 - 24z^4]$$

$$a_3 = (P)[(z/4) - (10/3)z^2 + z^3(11/3) - z^4(4/3)\ \ ]$$

$$a_4 = P[(3)z^2 - (7)z^3 + z^4(4)]$$

b_1=Pz/4

and $x_{10} = x_i = x_{20} = y_{11},\quad x_{11} = \alpha_{i+1},$

$x_{21} = x_{i+1},\quad x_{30} = x_i - h,\quad x_{31} = x_i + h,$

$x_{40} = x_{i+1} - h,\quad y_{41} = x_{i+1} + h,$

$y_{10} = x\ ,$

Thus

$$\sum_{i=1}^{4} a_i = \sum b_j = b_1 = zP = c(p,h)$$

Therefore, applying Lemma (4.3.1) for $m = 1$, $n = 4$ and $K = 1$, in (4.3.12) we get

$$|L_i^*(f)| \le k(P,h)\ w(f^{\{1\}},P) \tag{3.13}$$

and finally applying bounds of (3.11) and (3.13) in (3.12), we get inequality (3.5) of theorem 3.1.

IV DISSCUSTION AND RESULTS. We have discussed about discrete quartic spline and obtained error bounds and equations (3.4) and (3.5) are results in theorem 3.1

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