PRIME O – FILTER IN ALMOST DISTRIBUTIVE LATTICES

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Abstract : In this paper we derive some sufficient conditions for a prime filter of an Almost Distributive Lattice to become an O-filter. A set of equivalent conditions are established for every O-filter to become an annihilator filter. An equivalency is obtained between prime O-filters and maximal prime filters of an ADL.

***IndexTerms* - Prime O-filter, maximal prime filter, annihilator filter, ∗ - ADL.**

I. Introduction

After Booles axiomatization of two valued propositional calculus as a Boolean algebra, a number of generalizations both ring theoretically and lattice theoretically have come into being. The concept of an Almost Distributive Lattice (ADL) was introduced by Swamy and Rao as a common abstraction of many existing ring theoretic generalizations of a Boolean algebra on one hand and the class of distributive lattices on the other. In that paper, the concept of an filter in an ADL was introduced analogous to that in a distributive lattice and it was observed that the set PF(R) of all principal ideals of R forms a distributive lattice. This enables us to extend many existing concepts from the class of distributive lattices to the class of ADLs. Swamy, G. C. Rao and G. N. Rao introduced the concept of Dense element in ADLs. N. Rafi, Ravikumar, G. C. Rao introduced the concept of prime –o-ideals in ADLS. Sambasive Rao derived an equivalency is obtained between prime O - deals of a distributive lattice are co-maximal. In this paper we observe that the characterization of o-filters forms a distributive lattice. We also derive some sufficient conditions for a maximal prime filter of an ADL to become an O-filter.
In a *-ADL, a set of equivalent conditions are established for every o-filter to become an annihilator filter.

II. PRELIMINARY

2.1 Definition A Lattice is partially ordered set such that every pair of elements has a Least Upper Bound (Join) and Greatest Lower Bound (Meet).
The meet and join of element a, b are the denoted $a \vee b$ and $a \wedge b$ respectively.

2.2 Definition A partial ordered set (POSET) is a non – empty set together with a binary relation $\leq$ satisfying following axioms.

(i) $a \leq a$ (reflexive)
(ii) If $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetric)
(iii) If $a \leq b$ and $b \leq c$ then $a \leq c$ (transitive)

2.3 Definition A lattice L is said to be distributive. If

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

2.4 Definition An algebra $(L,\vee, \wedge, O)$ of type (2, 2, 0) is called an almost distributive lattice (ADL) if the following conditions hold

(i) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(ii) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(iii) $(a \vee b) \wedge a = a$
(iv) $(a \vee b) \wedge b = b$
(v) $a \vee (a \wedge b) = a$
(vi) $a \wedge a = a$

2.5 Definition Let $(L, \vee, \wedge, O)$ be an ADL with O. Then a unary operation $*$ on L is called a pseudo complement on L for all a, b $\in$ L. If

(i) $a \wedge a^* = 0$
(ii) $a \wedge b = 0 \Longrightarrow a^* \wedge b = b$
(iii) $(a \vee b)^* = a^* \wedge b^*$

Here a* is called a pseudo – complement of $a \in L$ and L is called pseudo complemented if a* exists for all $a \in L$. Moreover an element $a \in L$ is called dense if $a^* = 0$.

2.6 Definition For any filter F of an ADL L, define $O(F) = \{ x \in L / x \wedge f = 0 \text{ for some } f \in F\}$.

2.7 Definition A set of R is said to be multiplicatively closed subset of R if $S \neq \emptyset$ and for any a, b ∈ S, $a \wedge b \in S$.

2.8 Definition A non - empty subset I of R is said to be a Ideal if it satisfies,

(i) $a, b \in I \Longrightarrow a \vee b \in I$ and

(ii) $a \in I, x \in R \Longrightarrow a \wedge x \in I$

2.9 Definition A non - empty subset F of R is said to be a filter if it satisfies,

(i) $a, b \in F \Longrightarrow a \wedge b \in F$

(ii) $a \in F, x \in R \Longrightarrow x \vee a \in F$

2.10 Definition A prime filter G of R is said to be a maximal prime filter if there is no prime filter which properly contains the filter G.

2.11 Definition A prime ideal P of R is said to be a minimal prime ideal if there is no prime ideal which is properly contained in P.

III. CHARACTERIZATION OF O – FILTERS

3.1 Lemma Let F be a filters of an ADL L. If P is a maximal prime filter belonging to the filter O (F), then$P \cup F = \emptyset$.

Proof: Let P be a maximal prime filter belonging to the filter O (F). Suppose $x \in P \cup F$. Then $x \in P$ and $x \in F$. Since P is maximal, there exists $y \notin P$ such that $x \vee y \in O(F)$. Then $x \vee y \vee f = 0$ for some $f \in F$. Hence we get, $y \vee (x \vee f) = 0$ and $x \vee f \in F$. Thus $y \in O(F) \subseteq P$ which is a contradiction.

3.2 Lemma Every maximal prime filter of an ADL L belonging to an O – filter is a maximal prime filter in L.

Proof: Let F be an O – filter of L. Then we have F = O (I) for some ideal I of L. Let P be a maximal prime filter belonging to F = O (I). Then by the above lemma $P \cup F = \emptyset$. Let $x \in P$, then there exists $y \notin P$ such that $x \vee y \in O(I)$. Hence $x \vee y \vee f = 0$ for some $f \in I$. Thus $x \vee (y \vee f) = 0$ and $y \vee f \notin P$ (Since $P \cup I = \emptyset$ we get that $f \notin P$ and also $y \notin P$). Hence P is a maximal prime filter of L.

3.3 Theorem Every O – filter of an ADL L is the union of all maximal prime filters containing it.

Proof: Let F be an O – filter of an ADL L. Then F = O(I) for some ideal I of L. Let $F_0 = \cup\{$ P/ P is a maximal prime filter containing F$\}$. Clearly $F \subseteq F_0$. Conversely, Let $a \notin F = O(I)$, then $a \vee t \neq 0$ for all $t \in I$. Then there exists a maximal prime filter P such that $a \vee t \notin P$. Hence $a \notin P$ and $t \notin P$. Since P is prime, $[t)^* \subseteq P$ for all $t \in I$. Therefore F = O(I) $\subseteq P$. Thus P is a maximal prime filter containing F and $a \notin P$. We get $a \notin F_0$ which yield that $F_0 \subseteq F$. Therefore $F = F_0$.

3.4 Definition Let F be a filter of R. A prime filter P is said to be a maximal prime filter belonging to a filter F. If

(i) $F \subseteq P$ and

(ii) There is no prime filter Q such that $F \subseteq Q \subset P$

(ie) P is maximal among the prime filters of R containing F.

3.5 Lemma Every prime filter of R contains a maximal prime filter.

Proof: Let P be a prime filter of R. Let F = R – P. Then F is a prime ideal. Then there is a minimal prime ideal G in R. Now, $F \subseteq G \Longrightarrow R - G \subseteq R - F = P$. Therefore by the statement which is given below P is a maximal prime filter of R if and only if R – P is a minimal prime ideal of R. So that R – G is a maximal prime filter contained in P.

3.6 Definition A set S of R is said to be a multiplicatively closed subset of R if $S \neq \emptyset$ and for any a, b ∈ S and $a \vee b \in S$.

3.7 Lemma Let F be a filter and S be a multiplicatively closed subset of R such that $F \cup S = \emptyset$. Then there is a prime filter M of R such that $F \subseteq M$ and $M \cup S = \emptyset$.

Proof: Let A = {J/J is a filter of R, $F \subseteq J$ and $S \cup J = \emptyset$}. Since $\in A$, $A \neq \emptyset$. Clearly, A satisfies the hypothesis, therefore A has a maximal element M. Now, we prove M is prime. Let $a, b \in R$ and $a \notin M$, $b \in M$. Then $(M \wedge [a)) \cup S \neq \emptyset$ and $(M \wedge [b)) \cup S \neq \emptyset$. Let $x \in (M \wedge [a)) \cup S$ and $y \in (M \wedge [b)) \cup S$. Now, since S is a multiplicatively closed subset of R, $x \vee y \in S$ and $x \vee y \in (M \wedge [a)) \cup (M \wedge [b)) = M \wedge [a \vee b)$. If $a \vee b \in M$ then $x \vee y \in M \cup S$. Which is not true as $M \in A$. Therefore $a \vee b \notin M$. Thus M is prime.

IV. PRIME O – FILTERS IN ALMOST DISTRIBUTIVE LATTICE

4.1 Lemma For any almost distributive lattices L, for any prime filter F of O.

O (F) = O (L – F)

Proof: Let F be a prime filter of L. Then we have

$x \in O(F) \Leftrightarrow y \vee x = 0$ for some $y \notin F$

$\Leftrightarrow y \vee x = 0$ for some $y \in L - F$

$\Leftrightarrow x \in O(L - F)$.

Therefore O (F) = O (L –F) for any prime filter F of L.

4.2 Definition An ADL is called a $*$ - ADL, if to each $x \in L$, $[x)^{**} = [x')^{*}$ for some $x' \in L$.

4.3 Theorem Let L be an ADL. Then we have the following condition.

(a) Every maximal prime filter is an O – filter

(b) Every non – dense prime filter is an O – filter

Proof: (a) Let P be a maximal prime filter of L. Then L – P is a minimal ideal of L. Since P is maximal, We get P = O (P) = O(L – P). Hence P is an O – filter.

(b) Let P be a non - dense prime filter of L. Then there exists $O \neq x \in L$ such that $x \in P^{*}$.

Thus $P \subseteq P^{**} \subseteq [x)^{*}$. Let $a \in [x)^{*}$ then $a \vee x = 0 \in P$ and $x \notin P$. Because of $x \in P^{*}$. Hence $a \in P$. Then $[x)^{*} \subseteq P$. Hence we get $P = [x)^{*} = O\big((x]\big)$. Therefore P is an O – filter of L.

4.4 Theorem Let F be a proper O – filter of an almost distributive lattice L. Then F is filter if and only if F contains a prime filter.

Proof: First we have to assume that F contains a prime filter, say F. Since F is a O – filter of L. We get that F = O (I) for some ideal I of L. Choose a, b ∈ L such that a ∉ F and b ∉ F. Hence $b \vee a \notin F$. Thus $[b \vee a)^{*} \subseteq F = O(I)$. Suppose $b \vee a \in F = O(I)$. Then $b \vee a \vee I = 0$ for some $I \in I$. Hence $I \in I \cup O(I)$. Therefore $I \cup O(I) \neq \emptyset$. Thus F = O(I) = I = L. Which is contradiction. Therefore F is Prime filter.

4.5 Theorem Every prime O – filter of an almost distribution lattices L is a maximal prime filter.

Proof: Let F be a prime O – filter of L. Then F = O (I) for some ideal I of L. Let $x \in F = O(I)$. Then $x \vee y = 0$ for some $y \in I$, Suppose $y \in F$. Then $y \in I \cup O(I)$. Hence $I \cup O(I) \neq \emptyset$. By the condition, for any almost distributive lattice, for any ideal I of L, $I \cup O(I) \neq \emptyset \Longrightarrow I = O(I) = L,$ F = O (I) = I = L. Which is contradiction. Hence $y \notin F$. Therefore F is a maximal prime filter.

4.6 Definition A filter F of L is called an annihilator filter of L, if F = F** or equivalently F = S for some non – empty subset S of L.

4.7 Theorem Let L be a $*$ - ADL. Then the following conditions are equivalent (i) Every O – filter is an annihilator filter (ii) Every minimal prime filter is an annihilator filter(ii) Every prime O –filter is of the form $[x)^{**}$ for some $x \in L$

Proof: (i) $\Longrightarrow$ (ii) Obvious.

(ii) $\Longrightarrow$ (iii) Assume the condition (ii). Let P be a prime O – filter of L. Then by the above theorem P is a maximal prime filter. Hence by condition (ii), P is an annihilator filter. Thus P is a non – dense. Therefore P = $[x)^{*}$ for some non – zero $x \in P^{*}$. Since L is a $*$ - ADL, there exists $y \in L$ such that $P = [x)^{*} = [y)^{**}$.

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Thepreferredspellingoftheword "acknowledgment" inAmericaiswithoutan "e" afterthe "g".Avoidthestiltedexpression, "Oneofus(R.B.G.)thanks..." Instead,try"R.B.G.thanks".Putapplicablesponsoracknowledgmentshere;DONOTplacethemonthefirstpageofyourpaperorasafootnote.

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