PROOF OF KEPLER'S LAWS OF PLANETARY MOTION

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Abstract : Johannes Kepler was a German astronomer and mathematician who worked as an Imperial mathematician in Prague from 1601 to 1612. Using data collected by Tycho Brahe from naked-eye astronomical observations, he enacted three laws governing the motion of planets. Kepler began to study and observe the motion of Mars and, using his extraordinary arithmetic skills, had discovered the true orbital nature of Mars. They found that very few ellipses had actually gone. Such a discovery completely ended 2000 years of belief that the planets moved in a circle. These laws, especially the second (law of equal areas), provided strong evidence to Newton's law of universal gravitation. In this paper we proved the Kepler's laws of Planetary Motion.

Key words : Kepler's laws, Newton's law, universal gravitation, ellipse, foci.

Introduction

Johannes Kepler was a German astronomer and mathematician who worked as an Imperial mathematician in Prague from 1601 to 1612. Using data collected by Tycho Brahe from naked-eye astronomical observations, he enacted three laws governing the motion of planets. Kepler began to study and observe the motion of Mars and, using his extraordinary arithmetic skills, had discovered the true orbital nature of Mars by 1606. They found that very few ellipses had actually gone. Such a discovery completely ended 2000 years of belief that the planets moved in a circle. Kepler's three laws describe the observed motion of planets around the Sun.

Kepler also discovered that the planets do not move at the same speed throughout the ellipse. Instead, they move faster when they are closer to the sun. His results were published in a book called *Astronamia nova* (New Astronomy).

If Kepler's laws define the motion of the planets, Newton's laws define motion. Kepler's three laws describe the observed motion of planets around the Sun. They are equally valid in describing the motion of an artificial satellite around the Earth (ignoring any orbit disturbances for the time being). Knowledge of these laws, especially the second (law of equal areas), provided strong evidence to Sir Isaac Newton in 1684–85, when he outlined his famous law of gravity between the Earth and the Moon and between the Sun and the planets. There is validity for all goods in the universe. Newton showed that the motion of bodies subject to the central gravitational force does not always need to follow the elliptical orbits specified by Kepler's first law but can take paths defined by other, open conic curves, the motion can be in parabolic or hyperbolic orbits, depending on the total body energy. Thus, an object of sufficient energy, e.g., a comet - can enter the solar system and return unhindered. From Kepler's second law, it can be further observed that the angular momentum about any planet through the Sun and perpendicular to the orbital plane is also unchanged.

With the help of vector calculus, a later invention, Kepler's laws can be derived as consequences of Newton's inverse square law for gravitational attraction.

Kepler's Laws of Planetary Motion

Kepler's three laws describe the observed motion of the planets around the sun. They are equally valid in describing the motion of an artificial satellite around the earth (ignoring any orbit perturbations for the time being):

First Law (Law of Ellipses) : The orbit of a planet is an ellipse, with the sun at one focus.

Second Law (Law of Equal Areas) : The radius vector from the focus towards the satellite sweeps out equal areas in equal periods of time. Kepler's second law is really a statement of the conservation of angular momentum.

Third Law (Law of Distance) : The square of the orbital period of a planet is proportional to the cube of its mean distance to the focus.

An ellipse is a curve surrounding two points, called foci, so that the total distance from one focus to a point on the ellipse and back to the other focus is constant for every point on the curve.

NEWTON'S LAWS OF MOTION

Newton published the following three remarkably simple basic laws of motion, which together with his Law of Universal Gravitation form the physical basis for all theoretical work:

First Law Every object continues in its state of rest or of uniform motion in a straight line unless it is compelled to change that state by forces impressed upon it.

Second Law The rate of change of momentum measured relative to an inertial reference frame, is proportional to the force impressed and is in the same direction of that force.

Third Law To every action there is always opposed an equal reaction.

The second law can be expressed mathematically as follows:

$$F = m\frac{d^2\mathbf{r}}{dt^2}, \qquad (1)$$

where F is the vector sum of all forces acting on the mass m and $\frac{d^2\mathbf{r}}{dt^2}$ is the vector acceleration of the mass measured relative to an inertial reference frame.

NEWTON'S LAW OF UNIVERSAL GRAVITATION

In addition to stating the three laws of motion, Newton formulated the law of universal gravitation. Any two objects attract one another with a force proportional to the product of their masses and inversely proportional to the square of the distance between them. This law can be expressed mathematically in vector notation as

$$F = \frac{Gm_1m_2}{r^3}\mathbf{r} \qquad (2)$$

where F is the force on mass m_1 due to mass m_2. F is a vector in the direction from m_1 to m_2. G is the Universal Gravitational Constant.

PROOF OF KEPLER'S LAWS

The two-body problem was first stated and solved by Newton. The importance of this problem lies in two facts: Firstly, the problem involving spherical bodies in which the mass is distributed in spherical shells, is the only gravitational problem that can be solved rigorously (and relatively simply). Secondly, practical problems of orbital motion can be treated as approximate two-body problems, i.e. the two-body solution may be used to provide a first approximation of the orbital motion, and is therefore used as the starting point of the calculation of more accurate solutions.

The mathematical formulation of the two body problem results from a combination of equations (1) and (2). We choose the centre of the earth as the origin of our co-ordinate system, and we define the positive sense of the radius vector **r**, as the direction away from the origin. We first rewrite equations (1) and (2) so that they express the force acting on the satellite with mass m, due to the large mass M of the earth

$$F_m = m\frac{d^2\mathbf{r}}{dt^2}$$

$$F_m = -GmM\frac{\mathbf{r}}{r^3} \text{ (a minus sign because the force is towards the origin)} \qquad (1)$$

$$\frac{d^2\mathbf{r}}{dt^2} + GM\frac{\mathbf{r}}{r^3} = \frac{d^2}{dt^2}\mathbf{r} + \mu\frac{\mathbf{r}}{r^3} = 0 \qquad (3)$$

The integration of this basic equation, "The Two-Body Equation of Motion", is relatively straightforward for someone with a basic knowledge of Vector Analysis, and leads to the proof of Kepler's laws.

Proof of Constant Vector

Cross multiply equation (3) by **r**:

$$\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} + \frac{\mu}{r^3}\mathbf{r}\times\mathbf{r} = 0$$

The second term is zero since **r** × **r** = 0. Hence

$$\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} = 0 \qquad (4)$$

Take the derivative of the angular momentum h:

$$\frac{dh}{dt^2} = \frac{d}{dt^2}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt^2}\right) = \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} + \frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt} = 0 \qquad (5)$$

The first term on the right is zero because of (4). The second term is zero because the cross product of two equal vectors is zero. Hence: $\frac{dh}{dt} = 0,$ or in other words, h is a constant vector.

Proof of Kepler's First Law

Cross multiply equation (3) by **h**

$$\frac{d^2\mathbf{r}}{dt^2}\times\mathbf{h} = -\frac{\mu}{r^3}\mathbf{r}\times\mathbf{h} = -\frac{\mu}{r^3}\mathbf{r}\times\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt^2}\right)$$

Using the expression for the triple vector product, $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, the last expression becomes

$$-\frac{\mu}{r^3}\left[\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right)\mathbf{r} - (\mathbf{r}\cdot\mathbf{r})\frac{d\mathbf{r}}{dt}\right] = -\frac{\mu}{r^3}\left(r\frac{d\mathbf{r}}{dt}\mathbf{r} - r^2\frac{d\mathbf{r}}{dt}\right) = \mu\left(\frac{1}{r}\frac{d\mathbf{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\mathbf{r}\right) = \mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right),$$

Hence, $\frac{d^2\mathbf{r}}{dt^2}\times h = \mu\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$

This equation may be integrated directly, since h is constant:

$$\frac{d\mathbf{r}}{dt}\times h = \mu\left(\frac{\mathbf{r}}{r}\right) + c,$$

where c is a constant (vector) of integration. For reasons, which will become clear later, we write c as μe. The last expression then becomes:

$$\frac{d\mathbf{r}}{dt}\times h = \frac{\mu}{r}(\mathbf{r} + er) \quad (6)$$

Finally, take the dot product of eqn. (6) and **r**, using $a \cdot b \times c = c \cdot a \times b$

$$\mathbf{r}\cdot\left(\frac{d\mathbf{r}}{dt}\times h\right)=\frac{\mu}{r}\left(\mathbf{r}\cdot\mathbf{r}+e\cdot\mathbf{r}r\right),$$

or, $h\cdot\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=h\cdot h=h^2=\frac{\mu}{r}\left(r^2+er^2\cos v\right),$

where v is the angle between e and $\mathbf{r}$.

Hence, $r=\dfrac{h^2/\mu}{1+e\cos v}$ (7)

Equation (7) is the general equation in polar co-ordinates for a conic section with the origin at a focal point.

If $0 \Leftarrow e \Leftarrow 1$, the orbit is an ellipse that proofs Kepler's first law.

Proof of Kepler's Second Law

Rewrite the equation for h using $\mathbf{r}\cdot(\times h)\frac{d\mathbf{r}}{dt}=\frac{dr}{dt}e_r+r\frac{d\theta}{dt}e_\theta$ (from Fig. 1) (8)

$h=\mathbf{r}\times\frac{d\mathbf{r}}{dt}=\mathbf{r}\times\left(\frac{dr}{dt}e_r+r\frac{d\theta}{dt}e_\theta\right)=\frac{dr}{dt}\mathbf{r}\times e_r+r^2\frac{d\theta}{dt}e_r\times e_\theta$ (9)

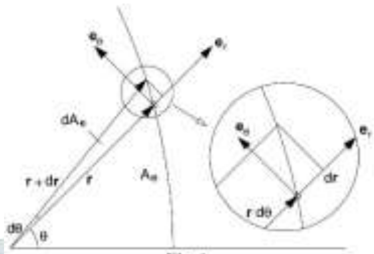

Fig. 1

In the above equation, the first term on the right is zero because $\mathbf{r}$ and e_r are collinear vectors.

Eqn. (8) can therefore be rewritten:

$h=r^2\frac{d\theta}{dt}$ (10)

Inspection of figure 1 shows that $r^2 d\theta=2dA_e$, (11)

i.e. twice the area swept by the radius vector per unit time. Since h is constant, it follows that the satellite sweeps out equal areas in equal periods of time, which proofs Kepler's second law.

Proof of Kepler's Third Law

The numerator of eqn. (7) is the semi-latus rectum p. Hence

$$\frac{h^2}{\mu}=p=a\left(1-e^2\right),$$

or, $h=\left[\mu a\left(1-e^2\right)\right]^{\frac{1}{2}}$ (12)

Combining equations (10) and (11), it follows that

$$h=2\frac{dA_e}{dt}$$

Since h is constant, it follows that $A_e(t)=\frac{h}{2}t=\frac{\left[\mu a\left(1-e^2\right)\right]^{\frac{1}{2}}}{2}t$ (13)

When $t = T$, or one orbital period, $A_e=\pi ab=\pi a^2\left(1-e^2\right)^{\frac{1}{2}}$ (14)

Combining eqn. (13) and (14), it follows that

$$\frac{\left[\mu a\left(1-e^2\right)\right]^{\frac{1}{2}}}{2}T=\pi a^2\left(1-e^2\right)^{\frac{1}{2}}$$

or, $T=2\pi\dfrac{a^{\frac{3}{2}}}{\mu^{\frac{1}{2}}}$ (15)

This proofs Kepler's third law.

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