Integral Solutions of the Diophantine equation

$$7x^2 - 4y^2 = 3z^2$$

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Abstract : The ternary quadratic homogeneous equation representing homogeneous cone given by $7x^2 - 4y^2 = 3z^2$ is analyzed for its non-zero distinct integer points on it. Different patterns of integer points satisfying the cone under consideration are obtained. The relations between the solutions and special number patterns are derived.

IndexTerms* - *Diophantine equation, integral solutions.

I. Introduction

Diophantine[2] equation is a polynomial equation in two or more unknowns There are different types of Diophantine equations and homogeneous Diophantine equation is one among them.The ternary cubic Diophantine equations are used in the research field to find their solutions. An infinite number of Diophantine equations are analysed for their integer solutions. This paper discusses different methods to find the integer solutions of the equation $7x^2-4y^2=3z^2$. Two different methods involving sub sections are discussed and the solutions are derived.

Notations:

- Triangular number of rank a is represented as[3]

$$t_{3,a} = \frac{a(a+1)}{2}$$

- Pentagonal pyramidal number of rank a is denoted by[1]

$$P_a^5 = \frac{a^2(a+1)}{2}$$

- Star number of rank a is represented as [4]

$$S_a = 6a(a-1)+1$$

- Tetrahedral number of rank a is denoted by [5]

$$P_a^3 = \frac{a(a+1)(a+2)}{6}$$

- Nasty number [6]

II. Method of Analysis:

The ternary quadratic equation under consideration is

$$7x^2 - 4y^2 = 3z^2 \qquad (1)$$

Different methods are employed to obtain non-zero distinct integer solutions to (1).

2.1 Method: 1

Using the linear transformations

$$x = R - 4T,\ y = R - 7T \qquad (2)$$

(1) becomes

$$R^2 = Z^2 + 28T^2 \qquad (3)$$

The solutions are given by

$$T = 2ab,\ z = a^2 - 28b^2,\ R = a^2 + 28b^2$$

In view of (2), the corresponding set of integer solutions to (1) is given below

$$x(a,b) = a^2 - 8ab + 28b^2$$

$$y(a,b) = a^2 - 14ab + 28b^2$$

$$z(a,b) = a^2 - 28b^2$$

where a,b are integers.

2.1.1 Properties:

- $x(a,a+1)-y(a,a+1)=12t_{3,a}$.
- $x(a,a(a+1))-y(a,a(a+1))=12P_a^5$.
- $x(a,a(a+1))-y(a,a(a+1))=12P_a^5$.
- $2(-8y(a,b)+14x(a,b)+6z(a,b))=$ Nasty number.
- $x(a,(a+1)(a+2))-y(a,(a+1)(a+2))=36P_a^3$.

2.1.2 Note: 1

Using the transformations ,

$$x=R+4T,\ y=R+7T \tag{4}$$

Another set of solutions are the given below

$$x(a,b)=a^2+8ab+28b^2$$

$$y(a,b)=a^2+14ab+28b^2$$

$$z(a,b)=a^2-28b^2$$

2.2 Method: 2

Equation (3) is equivalent to the system of double equations presented as below in Table 1

Table1 System of double equations

System	1	2	3	4	5	6
$R+z$	T^2	$28T$	$14T$	$7T$	$14T^2$	$7T^2$
$R-z$	28	T	$2T$	$4T$	2	4

Solving each of the above systems, the values of R, z and T are obtained. Substituting each of these values in (2), the corresponding x and y values are determined. For simplicity, the solutions obtained from the above systems are exhibited below in Table 2

Table 2 Solutions

System	Solutions (x,y,z)
1	$(2k^2-8k+14, 2k^2-14k+14, 2k^2-14)$
2	$(21k, 15k, 27k)$
3	$(4T, T, 6T)$
4	$(3k, -3k, 3k)$
5	$(7T^2-4T+1, 7T^2-7T+1, 7T^2-1)$
6	$(14k^2-8k+2, 14k^2-14k+2, 14k^2-2)$

2.2.1 Note: 2

By substituting the values of R, T, z from Table: II in (4), another set of the corresponding solutions are given in Table 3.

Table 3 Solutions

System	Solutions (x, y, z)
1	$(2k^2+8k+14, 2k^2+14k+14, 2k^2-14)$
2	$(37k, 43k, 27k)$
3	$(12T, 15T, 6T)$
4	$(19k, 25k, 3k)$
5	$(7T^2+4T+1, 7T^2+7T+1, 7T^2-1)$
6	$(14k^2+8k+2, 14k^2+14k+2, 14k^2-2)$

2.3 Method 3

(3) is written as

$$z^2+28T^2=R^2*1 \qquad (5)$$

Write 1 as

$$1=\frac{(3+i\sqrt{7})(3-i\sqrt{7})}{4^2} \qquad (6)$$

Assume $R=a^2+28b^2$ (7)

Using (7) and (6) in (5) and using the method of factorization, define

$$z+i2\sqrt{7}T=(a+i2\sqrt{7}b)^2\frac{(3+i\sqrt{7})}{4}$$

Equating the real and imaginary parts, we have

$$\left.\begin{array}{l} z=\frac{1}{4}(3a^2-84b^2-28ab) \\ T=\frac{1}{8}(a^2-28b^2+12ab) \end{array}\right\} \qquad (8)$$

For integer solutions, take $a=4A, b=2B$ in (8) and (7), we have

$$z=12A^2-84B^2-56AB \qquad (9)$$

$$\left.\begin{array}{l} T=2A^2-14B^2+12AB \\ R=16A^2+112B^2 \end{array}\right\} \qquad (10)$$

Substituting the above values of R and T in (2), the corresponding values of x and y are given by

$$\left.\begin{array}{l} x=8A^2+168B^2-48AB \\ y=2A^2+210B^2-84AB \end{array}\right\} \qquad (11)$$

Thus, (9) and (11) represent the integer solutions to (1).

2.3.1Note: 3

Substituting the above values of R, T, Z in (4), the corresponding values of x and y are

$$\left.\begin{array}{l} x=24A^2+56B^2+48AB \\ y=30A^2+14B^2+84AB \end{array}\right\} \qquad (12)$$

Thus, (9) and (12) represent the integer solutions to (1).

III. Remarks

- Since the considered equation (1) is symmetric in x, y, z, the following triples also satisfy (1):
  $(x, y, -z), (-x, -y, z), (-x, -y, -z)$,
  $(x, -y, -z), (-x, y, z)$.
- Equation (1) may be represented in the form of ratio as
  $$\frac{7(x-z)}{(y-z)} = \frac{4(y+z)}{x+z} = \frac{a}{b}, b \neq 0.$$

The corresponding solution are

$$x = a^2 - 8ab + 28b^2$$
$$y = -a^2 + 14ab - 28b^2$$
$$z = -a^2 + 28b^2$$

On performing calculations, it is observed that the values of x, y, z are now new

IV. Conclusion

Three different methods are adopted to find the integral solutions of the equation $7x^2-4y^2=3z^2$ and the relations between the solutions are also obtained.

V. References

[1] T.Aaron Gulliver, Sequences from Pentogonal Pyramids of integers, International Mathematical Forum,5(2010)No13, 621-628

[2] L.E. Dickson, History of Theory of Numbers, Vol.2, Chelsea Publishing company, NewYork, 1952.

[3] Pearl Joyce Berana, Jomari Motalbo ,Daryl Magpantay, On Triangular and Trapezoidal number,On Pacific Journal of Multidisciplinary Research, Vol 3, No4, Nov 2015 Part V.

[4] S.Vidhyalakshmi, A.Kavitha and M.A.Gopalan, Integral solution of the Heptic equation with five variables $x^3-y^3+(x^2+y^2)+z^3-w^3=2+11(x-y)p^6$,IJIRT,Vol1,Issue 1,2014,297-302.

[5] Maciei Ulas, On certain Diophantine equations related to Triangular and Tetrahedral number, Arxiv,15 (2008),1-8.

[6] V.Pandchelvi and P.Sivakama Sundari, Relations among some polygonal number and a nasty number of the form six times a perfect square, Int. Journal of Engineering Research and Applications, Vol 7,Issue 8, August 2017,12-15.

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