# FIXED POINT THEOREM FOR GENERALIZED WEAKLY COMPATIBLE MAPPINGS IN COMPLETE METRIC SPACES

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***Abstract*:** In this paper we prove common fixed point theorems for weakly compatible mappings in complete metric spaces by using a control function and implicit relation. Our results generalized and extend result of Mehta and Joshi [7].
**Keywords:** Common fixed point, control function, weakly compatible maps.

**AMS Subject classification (2000):** *47H10, 54H25*.

## I. INTRODUCTION

The weakly commuted mappings introduced by Sessa [12]. In1986 Jungck [5] generalized as compatible mappings of [12]. Pant [8] initiated the concept of $R$-weakly commuting mappings. Jungck and Rhoades [3] defined the weakly compatible mappings. Several mathematician [1, 2, 8 10, 11, 13] generalized weakly contractive mappings in Hilbert space. In 2001 Rhoades [10] generalized the concept of weakly contractive mappings in complete metric space.

Park [9] and Khan *et. al.* [6] proved fixed point theorems to the self mapping by using control function and altering distances between the points. Sastry [11] proved a common fixed point theorem for weakly commuting pairs of self mappings by using control function in complete metric space. A result to the fixed point obtained by Dutta and Choudhury [2] and generalized the concept of control function and weakly contractive mapping. Jungck [4] proved a common fixed point theorem by generalizing Banach's contraction principle to the commuting mappings.

The main aim of this paper is to present common fixed point results for weakly compatible mappings satisfying weak contractive condition by using control function in complete metric space.

## II. PRELIMINARY NOTES

Before proving our theorems we collect some definitions and results:

*Definition: 2.1.* A sequence $\{x_n\}$ in a metric space $(X,d)$ is said to be convergent to a point $x \epsilon X$, if $\lim_{n\to\infty} d(x_n, x) = 0$.

*Definition: 2.2.* A sequence $\{x_n\}$ in a metric space $(X,d)$ is said to be Couchy sequence if $\lim_{p\to\infty} d(x_n, x_m) = 0$. for all $n, m > p$.

*Definition: 2.3.* A metric space $(X,d)$ is called complete if every Couchy sequence in $X$ is convergent.

*Definition*: 2.4. Let $S$ and $T$ be self maps on $X$. maps $S$ and $T$ are called commuting if $STx = TSx$ for all $x \in X$.

*Definition:* 2.5. Let $S$ and $T$ be self maps on $X$. if $Sx = Tx$, for some $x \in X$ then $x$ is called coincidence point of $S$ and $T$.

*Definition: 2.6.* Let $S$ and $T$ be self maps defined on a set $X$. Then $S$ and $T$ are said to be weakly compatible if they commute at coincidence points. $i.e.$ If $Su = Tu$ for some $u \in X$, then $STu = TSu$.

*Definition: 2.7.* Let $S$ and $T$ be weakly compatible self mappings of a set $X$. If $S$ and $T$ have a unique point of coincidence, $i.e.$, If $w = Sx = Tx$ then $w$ is the unique common fixed point of $S$ and $T$.

*Definition: 2.8.* Let $S$ and $T$ be self mapping of nonempty subset $K$ of a metric space $X$. The mapping $S$ is called $T$ - contraction mapping, if there exists a real number $0 \le r < 1$ such that $d(Sx, Sy) \le r d(Tx, Ty)$ for all $x, y \in K$.

*Definition: 2.9[6].* A function $\varphi$ is defined as $\varphi: R^+ \to R^+$ which is continuous at zero, monotonically increasing and $\varphi(t) = 0$, If and only if $t = 0$.

*Definition: 2.10[1].* A self mapping $S$ of a metric space $(X,d)$ is said to be weakly contractive with respect to self mapping $T: X \to X$, If for each $x, y \in X$, $d(Sx, Sy) \le d(Tx, Ty) - \varphi(d(Tx, Ty))$,

Where $\varphi: [0,\infty) \to [0,\infty)$ is continuous non- decreasing function such that $\varphi$ is positive on $(0,\infty)$, $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$.

*Definition: 2.11[2].* Let $(X,d)$ be a complete metric space and $T: X \to X$, be a self mapping satisfying

$$\varphi(d(Tx, Ty)) \le \varphi(d(fx, fy)) - \phi(d(fx, fy)),$$

Where $\varphi, \phi: [0,\infty) \to [0,\infty)$, are both continuous and monotonic increasing functions with $\varphi(x) = 0 = \phi(x)$, if and only if $x = 0$.

Then $T$ has a unique fixed point.

*Theorem 2.12[7]* Let $(X,d)$ be a complete metric space. Suppose that $T$ and $f$ are self mappings of $X$.

Satisfies the following conditions:

*(i)* $T(X) \subseteq f(X)$ .

*(ii)* $T(X)$ is complete subspace of $X$.

*(iii)* $\varphi\left(d\,(Tx,Ty)\right) \leq \varphi\left(M\,(x,y)\right) - \phi\left(M\,(x,y)\right)$ Where $M\,(x,y) = F[d\,(Tx,fy), d\,(Ty,fx), d\,(Tx,fx), d\,(Ty,fy)]$ and $\varphi: [0,\infty) \rightarrow [0,\infty)$ is continuous and monotonic increasing function and $\phi: [0,\infty) \rightarrow [0,\infty)$ is continuous monotonic decreasing function such that $\varphi(x) = 0 = \phi(x)$ if and only if $x = 0$, $F \in F^*$.

*(iv)* The pair $(T,f)$ is weakly compatible.

Than $T$ and $f$ have a unique common fixed point.

**Implicit relations**

Suppose that $F^*$ be the set of continuous functions $F(t_1,t_2,t_3,t_4): [0,\infty)^4 \rightarrow [0,\infty)$ satisfying the following conditions:

$(F_1)$. $F$, is non decreasing in variable $t_1$.

$(F_2)$. For $u \geq 0$, $v \geq 0$; $F(u,u,v,0) \leq u$

$(F_3)$. $F(0,u,0,u) \leq u$ ; $F(u,0,0,u) \leq u$; $F(0,0,u,0) \leq u$; $F(u,u,u,0) \leq u$, for all $u > 0$.

## III. MAIN RESULTS

In this section presented a common fixed point theorem by using control function and implicit relation

*Theorem: 3.1*. Let $(X,d)$ be a complete metric space. Suppose that $A,B,S$ and $T$ are self mappings of $X$.

Satisfying the following conditions:

1. $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.
2. $A(X)$ and $B(X)$ are complete subspace of $X$.
3. $\varphi\left(d\,(Ax,By)\right) \leq \varphi\left(M\,(x,y)\right) - \phi\left(M\,(x,y)\right)$. Where $M\,(x,y) = F[d\,(Ax,Sx), d\,(Sx,Ty), d\,(By,Ty), d\,(Ax,Ty)]$ and $\varphi: [0,\infty) \rightarrow [0,\infty)$ is continuous and monotonic increasing function and $\phi: [0,\infty) \rightarrow [0,\infty)$ is continuous and monotonic decreasing function such that $\varphi(x) = 0 = \phi(x)$ if and only if $x = 0$, $F \in F^*$.
4. $(A,S)$ and $(B,T)$ are weakly compatible.

Than $A,B,S$ and $T$ have a unique common fixed point.

**Proof:** Let $x_0 \in$ X and $\{y_n\}$ is a sequence of $X$ such that

$$y_n = Ax_n = Tx_{n+1};\ \text{and}\ , y_{n+1} = Bx_{n+1} = Sx_{n+2}\ \text{for every}\ n \geq 0.$$

By condition (3) we have,

$$\varphi(d(y_n,y_{n+1})) = \varphi\left(d\,(Ax_n,Bx_{n+1})\right) \leq \varphi\left(M\,(x_n,x_{n+1})\right) - \phi\left(M\,(x_n,x_{n+1})\right). \tag{3.1.1}$$

Where $M\,(x_n,x_{n+1}) = F[\,d\,(Ax_n,Sx_n), d\,(Sx_n,Tx_{n+1}), d\,(Bx_{n+1},Tx_{n+1}), d\,(Ax_n,Tx_{n+1})]$

$$= F[d\,(Ax_n,Bx_{n-1}), d\,(Bx_{n-1},Ax_n), d\,(Bx_{n+1},Ax_n), d\,(Ax_n,Ax_n)]$$

$$= F[d\,(y_n,y_{n-1}), d\,(y_{n-1},y_n), d\,(y_{n+1},y_n), d\,(y_n,y_n)]$$

$$= F[d\,(y_n,y_{n-1}), d\,(y_n,y_{n-1}), d\,(y_n,y_{n+1}), 0\,]$$

$$\Rightarrow M\,(x_n,x_{n+1}) \leq d\,(y_n,y_{n-1})$$

From (3.1.1) we have

$$\varphi(d(y_n,y_{n+1})) \leq \varphi\left(d\,(y_n,y_{n-1})\right) - \phi\left(d\,(y_n,y_{n-1})\right). \tag{3.1.2}$$ ⇒

$$\varphi(d(y_n,y_{n+1})) \leq \varphi\left(d\,(y_n,y_{n-1})\right)$$

Since $\varphi$ is monotonic increasing function.

Therefore the sequence $\{d(y_n,y_{n+1})\}$ is monotonic decreasing.

$$\therefore \lim_{n\to\infty} d(y_n,y_{n+1}) = r,$$

Where $r$ is a non negative real number.

Letting, $n \rightarrow \infty$, then by equation (3.1.2) we have

$$\varphi(r) \leq \varphi\,(r) - \phi(r)$$

$$\Rightarrow \phi(r) \leq 0,$$

which is possible only if $r = 0$.

Thus $\lim_{n\to\infty} d(y_n,y_{n+1}) = 0$.

Now we prove that $\{y_n\}$ is a Cauchy sequence.

If possible let $\{y_n\}$ is not a Cauchy sequence.

Then there exist $\varepsilon > 0$ for which we can find subsequence $\{y_{m_k}\}$ and $\{y_{n_k}\}$ of $\{y_n\}$ such that $n_k$ is a smallest index for which

$$n_k > m_k > k,$$

$$d(y_{m_k}, y_{n_k}) \geq \boldsymbol{\varepsilon}\ \text{and}\ d\left(y_{m_k}, y_{n_k-1}\right) < \boldsymbol{\varepsilon}. \tag{3.1.3}$$

So we have

$$\boldsymbol{\varepsilon} \leq d(y_{m_k}, y_{n_k}) \leq d\left(y_{m_k}, y_{n_k-1}\right) + d\left(y_{n_k-1}, y_{n_k}\right) < \boldsymbol{\varepsilon} + d\left(y_{n_k-1}, y_{n_k}\right)$$

Taking $k \rightarrow \infty$ and using $d\left(y_n,y_{n+1}\right) \rightarrow 0$

We have

$$\lim_{n\to\infty} d(y_{m_k}, y_{n_k}) = \boldsymbol{\varepsilon}. \tag{3.1.4}$$

Now we have

$$d(y_{m_k}, y_{n_k}) \le d(y_{m_k}, y_{m_k-1}) + d(y_{m_k-1}, y_{n_k-1}) + d(y_{n_k-1}, y_{n_k})$$
$$d(y_{m_k-1}, y_{n_k-1}) \le d(y_{m_k-1}, y_{m_k}) + d(y_{m_k}, y_{n_k}) + d(y_{n_k}, y_{n_k-1})$$

Taking $k \to \infty$, we have $\lim_{n\to\infty} d(y_{m_k-1}, y_{n_k-1}) = \boldsymbol{\varepsilon}$.

Now by condition (3) and (3.1.3) we have

$$\varphi(\varepsilon) \le \varphi\left(d(y_{m_k}, y_{n_k})\right) \le \varphi\left(M(x_{m_k}, x_{n_k})\right) - \phi\left(M(x_{m_k}, x_{n_k})\right). \quad (3.1.5)$$

Where $M(x_{m_k}, x_{n_k}) = F[\, d\,(Ax_{m_k}, Sx_{m_k}), d\,(Sx_{m_k}, Tx_{n_k}), d\,(Bx_{n_k}, Tx_{n_k}), d\,(Ax_{m_k}, Tx_{n_k})]$

$$= F[\, d\,(Ax_{m_k}, Bx_{m_k-1}), d\,(Bx_{m_k-1}, Ax_{n_k-1}), d\,(Bx_{n_k}, Ax_{n_k-1}), d\,(Ax_{m_k}, Ax_{n_k-1})]$$
$$= F[\, d\,(y_{m_k}, y_{m_k-1}), d\,(y_{m_k-1}, y_{n_k-1}), d\,(y_{n_k}, y_{n_k-1}), d\,(y_{m_k}, y_{n_k-1})]$$

Taking $k \to \infty$, $\lim_{k\to\infty} M(x_{m_k}, x_{n_k}) = F[0, \boldsymbol{\varepsilon}, 0, \boldsymbol{\varepsilon}]$

Thus by, $(F_3)$ we have $\lim_{k\to\infty} M(x_{m_k}, x_{n_k}) \le \boldsymbol{\varepsilon}$. (3.1.6)

Therefore by (3.1.5) and (3.1.6) we get, $\varphi(\varepsilon) \le \varphi(\varepsilon) - \phi(\varepsilon$

$$\Rightarrow \phi(\varepsilon) \le 0.$$

Which is a contradiction because as $\phi: [0, \infty) \to [0, \infty)$,

We get $\phi(\varepsilon) \ge 0$ and $\phi(\varepsilon) = 0$ if and only if, $\varepsilon = 0$.

Hence our assumption is wrong.

Thus $\{y_n\}$ is a Cauchy sequence in $A(X)$.

Since $A(X)$ is a complete subspace of $X$,

Therefore the sequence $\{y_n\}$ is converges in $X$.

Therefore there exists $z \in X$ such that $\lim_{n\to\infty} \{y_n\} = z$.

Thus $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_{n+1} = z$ and $\lim_{n\to\infty} Bx_{n+1} = \lim_{n\to\infty} Sx_{n+2} = z$

i.e., $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Tx_{n+1} = \lim_{n\to\infty} Bx_{n+1} = \lim_{n\to\infty} Sx_{n+2} = z$. (3.1.7)

Since $B(X) \subseteq S(X)$, there exists a point $u \in X$ such that $z = Su$.

From condition (3) we have $\varphi(d(Au, z)) \le \varphi(d(Au, Bx_{n+1})) + \varphi(d(Bx_{n+1}, z))$

$$\Rightarrow \varphi(d(Au, z)) \le \varphi\left(M\,(u, x_{n+1})\right) - \phi\left(M\,(u, x_{n+1})\right) + \varphi(d(Bx_{n+1}, z)). \quad (3.1.8)$$

Where $M\,(u, x_{n+1}) = F[\, d\,(Au, Su), d\,(Su, Tx_{n+1}), d\,(Bx_{n+1}, Tx_{n+1}), d\,(Au, Tx_{n+1})]$

$$\Rightarrow M\,(u, x_{n+1}) = F[\, d\,(Au, z), d\,(z, Tx_{n+1}), d\,(Bx_{n+1}, Tx_{n+1}), d\,(Au, Tx_{n+1})]. \quad (3.1.9)$$

Taking $n \to \infty$, we have

$$M\,(u, x_{n+1}) = F[\, d\,(Au, z), d\,(z, z), d\,(z, z), d\,(Au, z))]$$
$$= F[\, d\,(Au, z), 0, 0, d\,(Au, z))]$$

By using, $(F_3)$, we get $M\,(u, x_{n+1}) \le d\,(Au, z)$. (3.1.10)

From (3.1.8) and (3.1.10), we have

$$\varphi(d(Au, z)) \le \varphi\left(d\,(Au, z)\right) - \phi\left(d\,(Au, z)\right) + \varphi(d(z, z))$$
$$\Rightarrow \phi\left(d\,(Au, z)\right) \le 0.$$

Which is possible only if $Au = z$.

Thus, $Su = Au = z$, so $u$ is a coincidence point of $A$ and $S$.

But the pair $A$ and $S$ are weakly compatible, $ASu = SAu$, i.e., $Az = Sz$.

Again since $A(X) \subseteq T(X)$, there exists a point $v \in X$ such that $z = Tv$.

From condition (3) we have

$$\varphi(d(z, Bv)) = \varphi(d(Au, Bv))$$
$$\le \varphi\left(M\,(u, v)\right) - \phi\left(M\,(u, v)\right). \quad (3.1.11)$$

Where $M\,(u, v) = F[\, d\,(Au, Su), d\,(Su, Tv), d\,(Bv, Tx_{n+1}), d\,(Au, Tv)]$

$$= F[\, d\,(z, z), d\,(z, Tv), d\,(Bv, Tv), d\,(Au, Tv)]$$
$$= F[\, 0, 0, d\,(Bv, z), 0)]$$

By using,$(F_3)$, we get $M\,(u, v) \le d\,(Bv, z)$. (3.1.12)

From (3.1.11) and (3.1.12), we have

$$\varphi(d(z, Bv)) \le \varphi\left(d\,(z, Bv)\right) - \phi\left(d\,(z, Bv)\right)$$
$$\Rightarrow \phi\left(d\,(z, Bv)\right) \le 0.$$

Which is possible only if $Bv = z$.

Hence $Tv = Bv = z$, so $v$ is a coincidence point of $B$ and $T$.

But the pair $B$ and $T$ are weakly compatible, $BTv = TBv$, i.e., $Bz = Tz$.

Now we show that $z$ is a fixed point of $A$.

By condition (3) we have

$$\varphi(d(Az, z)) = \varphi(d(Az, Bv)) \le \varphi\left(M\,(z, v)\right) - \phi\left(M\,(z, v)\right). \quad (3.1.13)$$

Where $M\,(z, v) = F[\, d\,(Az, Sz), d\,(Sz, Tv), d\,(Bv, Tz), d\,(Az, Tv)]$

$$= F[\, d\,(Az, Sz), d\,(Sz, z), d\,(Az, z))]$$
$$= F[\, d\,(Az, Az), d\,(Az, z), 0,\, d\,(Az, z)]$$

By using $(F_3)$ we get

$$M\,(z, v) \le d\,(Az, z). \quad (3.1.14)$$

From (3.1.13) and (3.1.14) we have

$$\varphi(d(Az, z)) \le \varphi\left(d\,(Az, z)\right) - \phi\left(d\,(Az, z)\right)$$
$$\Rightarrow \phi\left(d\,(Az, z)\right) \le 0$$

Which is possible only if $Az = z$

Hence $Az = Sz = z$
Now we show that $z$ is a fixed point of $B$.
By condition (3) we have

$$\varphi(d(z,Bz)) = \varphi(d(Az,Bz)) \leq \varphi(M(z,z)) - \phi(M(z,z)). \quad (3.1.15)$$

Where $M(z,z) = F[d(Az,Sz), d(Sz,Tz), d(Bz,Tz), d(Az,Tz)]$
$= F[0, d(z,Bz), 0, d(z,Bz)] = F[0, d(z,Bz), 0, d(z,Bz)]$

By using,$(F_3)$ we get

$$M(z,z) \leq d(z,Bz). \quad (3.1.16)$$

From (3.1.15) and (3.1.16) we have

$$\varphi(d(z,Bz)) \leq \varphi(d(z,Bz)) - \phi(d(z,Bz))$$
$$\Rightarrow \phi(d(z,Bz)) \leq 0$$

Which is possible only if $Bz = z$
Hence $Bz = Tz = z$
Therefore $Az = Sz = Bz = Tz = z, i.e., z$ is a common fixed point of $A, B, S$ and $T$.
**Uniqueness:** If possible let $w$ be another common fixed point of $A, B, S$ and $T$.
By condition (3) we have

$$\varphi(d(z,w)) = \varphi(d(Az,Bw)) \leq \varphi(M(z,w)) - \phi(M(z,w)). \quad (3.1.17)$$

Where $M(z,w) = F[d(Az,Sz), d(Sz,Tw), d(Bw,Tw), d(Sw,Bw), d(Az,Tw)]$
$= F[d(z,z), d(z,w), d(w,w), d(z,w)]$
$= F[0, d(z,w), 0, d(z,w)]$

By using,$(F_3)$ we get

$$M(z,w) \leq d(z,w). \quad (3.1.18)$$

From (3.1.17) and (3.1.18) we have,

$$\varphi(d(z,w)) \leq \varphi(d(z,w)) - \phi(d(z,w))$$
$$\Longrightarrow \phi(d(z,w)) \leq 0$$

Which is possible only if $z = w$.
Hence $z$ is the unique common fixed point of $A, B, S$ and $T$.
***Corollary: 3.2***. Let $(X, d)$ be a complete metric space. Suppose that $A, B$ and $S$ are self mappings of $X$
Satisfying the following conditions:

1. $A(X) \subseteq S(X)$ and $B(X) \subseteq S(X)$.
2. $A(X)$ and $B(X)$ are complete subspace of $X$.
3. $\varphi(d(Ax,By)) \leq \varphi(M(x,y)) - \phi(M(x,y))$
   Where $M(x,y) = F[d(Ax,Sx), d(Sx,Sy), d(By,Sy), d(Ax,Sy)]$ and $\varphi: [0,\infty) \rightarrow [0,\infty)$ is continuous and monotonic increasing function and $\phi: [0,\infty) \rightarrow [0,\infty)$ is continuous monotonic decreasing function such that $\varphi(x) = 0 = \phi(x)$ if and only if $x = 0$, $F \in F^*$.
4. The pair $(A, S)$ and $(B, S)$ are weakly compatible.

Than $A, B$ and $S$ have a unique common fixed point.
**Proof:** By taking $T = S$ in theorem 3.1 we get the proof.
***Corollary: 3.3***. Let $(X, d)$ be a complete metric space. Suppose that $A$ and $S$ are self mappings of $X$
Satisfying the following conditions:

1. $A(X) \subseteq S(X)$.
2. $A(X)$ is complete subspace of $X$.
3. $\varphi(d(Ax,Ay)) \leq \varphi(M(x,y)) - \phi(M(x,y))$
   Where $M(x,y) = F[d(Ax,Sx), d(Sx,Sy), d(Ay,Sy), d(Ax,Sy)]$ and $\varphi: [0,\infty) \rightarrow [0,\infty)$ is continuous and monotonic increasing function and $\phi: [0,\infty) \rightarrow [0,\infty)$ is continuous monotonic decreasing function such that $\varphi(x) = 0 = \phi(x)$ if and only if $x = 0$, $F \in F^*$.
4. The pair $(A, S)$ is weakly compatible.

Than $A$ and $S$ have a unique common fixed point.
**Proof:** By taking $B = A$ and $T = S$ in theorem 3.1 we get the proof.

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