A common fixed point theorem using common E.A. like property in fuzzy 2-metric space

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Abstract **:** The aim of the present paper is to prove a common fixed point theorem for four mappings satisfying common E.A. like property in fuzzy 2-metric spaces, which generalize and improve the result of Subhani and Kumar [13].

Keywords: Fuzzy 2-metric spaces, weakly compatible mapping, common E.A. like property.

I. INTRODUCTION

In 1975, Kramosil and Michalek [7] introduced the concept of fuzzy metric space using the concept of [18], which opened an avenue for further development of analysis in such spaces. George and Veeramani [5] modified the notion of fuzzy metric spaces with the help of t-norm in 1994.

On the other hand, Aamri and El. Moutawakil [1] generalized the concepts of non-compatibility by defining the notion of E.A. property in metric space. Pant and Pant [10] defined the same property in fuzzy metric space. The concept of weakly compatible maps by [6] in fuzzy metric space is generalized by A.Al Thagafi and Shahzad [2] by introducing the concept of occasionally weakly compatible mappings. Gähler investigated 2-metric spaces in his papers [4]. Recently, Yadav and Thakur [17] generalized the result of Vasuki [14] for fuzzy 2-metric spaces and proved a common fixed point theorem for R-weakly commuting self mappings. Wadhwa et al. [16] introduced the notion of common E.A. like property and proved some common fixed point theorems in fuzzy metric spaces. Wadhwa and Bhardwaj [15] improved and generalized the result of [17] and proved a common fixed point theorem for common E.A. like property in fuzzy 2-metric space.

In this paper we prove a common fixed point theorem for four mappings satisfying common E.A. like property in fuzzy 2-metric spaces, which generalize and improve the result of Subhani and Kumar [13] in the light of [3].

II. PRELIMINARY

Definition 2.1 [11]: An operation *: $[0.1]3 \rightarrow [0, 1]$ is called a t-norm of {[0.1], *} is an abelian topological monoid with unit 1 such that $a1 * b1 * c1 \leq a2 * b2 * c2$ whenever $a1 \leq a2$, $b1 \leq b2$, $c1 \leq c2$, $\forall$ a1, b1, c1, a2, b2, c2 $\in [0, 1]$.

Definition 2.2 [17]: A 3-taple (X, M, *) is said to be a fuzzy 2- metric space if X is an arbitrary set, * is a continuous t-norm and M is a fuzzy set on $X3 \times [0, \infty)$ satisfying the following conditions; $\forall$ x, y, z $\in$X, s, t > 0

1) M(x, y, z, 0) = 0,

2) M (x, y, z, t) = 1 for all t > 0 if and only if at least two of three points are equal,

3) M (x, y, z, t) = M (y, x, z, t) = M (z, x, y, t) symmetry about three variables,

4) $M(x, y, u, t1) * M(x, u, z, t2) * M(u, y, z, t3) \leq M(x, y, z, t1+t2+t3)$

$\forall$ x, y, z, u$\in$X and t1, t2, t3>0,

5) $M(x, y, z, \cdot): [0, \infty) \rightarrow [0, 1]$ is left continuous,

6) $\lim t \rightarrow \infty$ M(x, y, z, t) = 1.

The function value M(x, y, z, t) may be interpreted as the probability that the area of triangle is less than t.

Definition 2.3 [12]: A pair of self mapping {f, g} of a fuzzy 2-metric space (X, M, *) is said to be weakly compatible if they commute at the coincidence point i.e., If fu = gu for some u $\in$X, then fgu=gfu.

Definition 2.4 [10]: Let f and g be two self-maps of a fuzzy metric space (X, M, *) then they are said to satisfy E.A property if there exists a sequence {xn} in X such that

limn→∞fxn = limn→∞gxn = z for some z ∈X.

Definition 2.5 [12]: A pair of self-mapping {f, g} of a fuzzy 2-metric spaces (X, M, *) is said to satisfy E.A property, if there exists a sequence {xn} in X such that

limn→∞ M(fxn, gxn, a, t) = t for some t∈X

Definition 2.6 [16]: Let A, B, S and T be self maps of a fuzzy metric space (X, M, *), then the pairs (A, S) and (B, T) said to satisfy common E.A. like property if there exists two sequences {xn} and {yn} in X such that

lim n→∞Axn = lim n→∞Sxn = lim n→∞Tyn = lim n→∞Byn = z,

where z ∈ S(X) ∩ T(X) or z ∈ A(X) ∩ B(X).

Lemma 2.7 [15]: Let (X, M, *) be a fuzzy 2-metric space. If there exists k∈(0, 1) such that

M(x, y, z, kt) ≥ M(x, y, z, t) for all x, y, z∈X with z≠x, z≠y and t>0, then x=y.

Subhani and Kumar [13] proved following result:

Theorem 2.8 [13]: Let (X, M, *) be a complete fuzzy metric space and let A, B, S and T be self mappings of X. Let the pairs (A, S) and (B, T) are OWC. If there exists a point q∈(0, 1), for all x, y∈X and t > 0, such that

$$M(Ax,By,z,qt) \geq \alpha_1 \min\begin{Bmatrix} M(Sx,Ty,t), \\ M(Sx,Ax,t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(By,Ty,t), \\ M(Ax,Ty,t) \end{Bmatrix}$$

$$+ \alpha_3\, M(By,Sx,t) + \alpha_4\, M(Ax,By,t)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, and $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 1$.

Then there exists a unique point of $w \in X$, such that $Aw=Sw=w$ and a unique point $z \in X$, such that $Bz=Tz=z$. Moreover $z=w$, so that there is a unique common fixed point of A,B,S and T.

3. Main Result

Theorem 3.1: Let A, B, S and T be self maps of a fuzzy 2-metric spaces (X, M, *) satisfying the following condition:

(3.1.1) if there exists a constant k∈(0, 1), for all x, y, z∈X and t > 0, such that

$$M(Ax,By,z,qt) \geq \alpha_1 \min\begin{Bmatrix} M(Sx,Ty,z,t), \\ M(Sx,Ax,z,t), \\ M(Ax,By,z,t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(By,Ty,z,t), \\ M(Ax,Ty,z,t) \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} M(By,Sx,z,t), \\ M(Ax,By,z,t) \end{Bmatrix} + \alpha_4\, M(Ax,By,z,t)$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, and $(\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) > 0$,

(3.1.2) pairs (A, S) and (B, T) are weakly compatible and satisfy common E.A. like property, then A, B, S and T have a unique common fixed point in X.

Proof: Since (A, S) and (B, T) satisfy common E. A. Like property therefore there exist two sequences {xn} and {yn} in X such that

limn→∞Axn = limn→∞Sxn = limn→∞Tyn = limn→∞Byn = z1

where z1 ∈S(X)∩T(X) or z1 ∈A(X)∩B(X).

Suppose z1 ∈S(X)∩T(X), now we have limn→∞Axn = z1 ∈ S(X) then z1 = Su for some u ∈ X.

No we claim that Au = Su, from (3.1.1) we have,

$$M(Au, By_n, z, qt) \geq \alpha_1 \min\begin{Bmatrix} M(Su, Ty_n, z, t), \\ M(Su, Ax, z, t), \\ M(Au, By_n, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(By_n, Ty_n, z, t), \\ M(Au, Ty_n, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} M(By_n, Su, z, t), \\ M(Au, By_n, z, t) \end{Bmatrix} + \alpha_4 M(Au, By_n, z, t)$$

Taking limit $n \to \infty$, we get

$$M(Au, z_1, z, qt) \geq \alpha_1 \min\begin{Bmatrix} M(z_1, z_1, z, t), \\ M(z_1, Ax, z, t), \\ M(Au, z_1, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(z_1, z_1, z, t), \\ M(Au, z_1, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} M(z_1, z_1, z, t), \\ M(Au, z_1, z, t) \end{Bmatrix} + \alpha_4 M(Au, z_1, z, t)$$

$$M(Au, z_1, z, qt) \geq \alpha_1 \min\begin{Bmatrix} 1, \\ M(z_1, Ax, z, t), \\ M(Au, z_1, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} 1, \\ M(Au, z_1, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} 1, \\ M(Au, z_1, z, t) \end{Bmatrix} + \alpha_4 M(Au, z_1, z, t)$$

$$M(Au, z_1, z, qt) \geq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) M(Au, z_1, z, t) > M(Au, z_1, z, t)$$

$M(Au, z_1, z, kt) \geq M(Au, z_1, z, t)$;

Lemma 2.7 implies that $Au = z_1 = Su$.

Since the pair (A, S) is weak compatible, therefore $Az_1 = ASu = SAu = Sz_1$.

Again, $\lim_{n\to\infty} By_n = z_1 \in T(X)$ then $z_1 = Tv$ for some $v \in X$.

No we claim that Tv = Bv, from (3.1.1) we have,

$$M(Ax_n, Bv, z, qt) \geq \alpha_1 \min\begin{Bmatrix} M(Sx_n, Tv, z, t), \\ M(Sx_n, Ax_n, z, t), \\ M(Ax_n, Bv, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(By, Tv, z, t), \\ M(Ax_n, Tv, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} M(By, Sx_n, z, t), \\ M(Ax_n, Bv, z, t) \end{Bmatrix} + \alpha_4 M(Ax_n, Bv, z, t)$$

Taking limit $n \to \infty$, we get,

$$M(z_1, Bv, z, qt) \geq \alpha_1 \min\begin{Bmatrix} M(z_1, z_1, z, t), \\ M(z_1, z_1, z, t), \\ M(z_1, Bv, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(Bv, z_1, z, t), \\ M(z_1, z_1, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} M(Bv, z_1, z, t), \\ M(z_1, Bv, z, t) \end{Bmatrix} + \alpha_4 M(z_1, Bv, z, t)$$

$$M(z_1, Bv, z, qt) \geq \alpha_1 \min\begin{Bmatrix} 1, \\ 1, \\ M(z_1, Bv, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(Bv, z_1, z, t), \\ 1 \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} M(Bv, z_1, z, t), \\ M(z_1, Bv, z, t) \end{Bmatrix} + \alpha_4 M(z_1, Bv, z, t)$$

$$M(z_1, Bv, z, qt) \geq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) M(z_1, Bv, z, t) > M(z_1, Bv, z, t)$$

$$M(z_1, Bv, z, kt) \geq M(z_1, Bv, z, t);$$

Lemma 2.7 implies that $Bv = z_1 = Tv$.

Since the pair (B, T) is weak compatible, therefore $Tz_1 = TBv = BTv = Bz_1$.

Now we show that $Az_1 = z_1$, from (3.1.1) we have,

$$M(Az_1, By_n, z, qt) \geq \alpha_1 \min\begin{Bmatrix} M(Sz_1, Ty_n, z, t), \\ M(Sz_1, Az_1, z, t), \\ M(Az_1, By_n, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(By_n, Ty_n, z, t), \\ M(Az_1, Ty_n, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} M(By_n, Sz_1, z, t), \\ M(Az_1, By_n, z, t) \end{Bmatrix} + \alpha_4 M(Az_1, By_n, z, t)$$

Taking limit n → ∞, we get

$$M(Az_1, z_1, z, qt) \geq \alpha_1 \min\begin{Bmatrix} M(z_1, z_1, z, t), \\ M(z_1, Az_1, z, t), \\ M(Az_1, z_1, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(z_1, z_1, z, t), \\ M(Az_1, z_1, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} M(z_1, z_1, z, t), \\ M(Az_1, z_1, z, t) \end{Bmatrix} + \alpha_4 M(Az_1, z_1, z, t)$$

$$M(Az_1, z_1, z, qt) \geq \alpha_1 \min\begin{Bmatrix} 1, \\ M(z_1, Az_1, z, t), \\ M(Az_1, z_1, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} 1, \\ M(Az_1, z_1, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} 1, \\ M(Az_1, z_1, z, t) \end{Bmatrix} + \alpha_4 M(Az_1, z_1, z, t)$$

$$M(Az_1, z_1, z, qt) \geq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) M(Az_1, z_1, z, t) > M(Az_1, z_1, z, t)$$

$$M(Az_1, z_1, z, kt) \geq M(Az_1, z_1, z, t);$$

Lemma 2.7 implies that $Az_1 = z_1$.

Now we show that $Bz_1 = z_1$, from (3.1.1) we have,

$$M(Ax_n, Bz_1, z, qt) \geq \alpha_1 \min\begin{Bmatrix} M(Sx_n, Tz_1, z, t), \\ M(Sx_n, Ax_n, z, t), \\ M(Ax_n, Bz_1, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(Bz_1, Tz_1, z, t), \\ M(Ax_n, Tz_1, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} M(Bz_1, Sx_n, z, t), \\ M(Ax_n, Bz_1, z, t) \end{Bmatrix} + \alpha_4 M(Ax_n, Bz_1, z, t)$$

Taking limit n → ∞, we get,

$$M(z_1, Bz_1, z, qt) \geq \alpha_1 \min\begin{Bmatrix} M(z_1, z_1, z, t), \\ M(z_1, z_1, z, t), \\ M(z_1, Bz_1, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(Bz_1, z_1, z, t), \\ M(z_1, z_1, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} M(Bz_1, z_1, z, t), \\ M(z_1, Bz_1, z, t) \end{Bmatrix} + \alpha_4 M(z_1, Bz_1, z, t)$$

$$M(z_1, Bz_1, z, qt) \geq \alpha_1 \min\begin{Bmatrix} 1, \\ 1, \\ M(z_1, Bz_1, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(Bz_1, z_1, z, t), \\ 1 \end{Bmatrix}$$

$$+ \alpha_3 \min\begin{Bmatrix} M(Bz_1, z_1, z, t), \\ M(z_1, Bz_1, z, t) \end{Bmatrix} + \alpha_4 M(z_1, Bz_1, z, t)$$

$$M(z_1, Bz_1, z, qt) \geq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) M(z_1, Bz_1, z, t) > M(z_1, Bz_1, z, t)$$

$$M(z_1, Bz_1, z, kt) \geq M(z_1, Bz_1, z, t);$$

Lemma 2.7 implies that $Bz_1 = z_1$.

Hence, $Az_1 = Sz_1 = Bz_1 = Tz_1 = z_1$.

Thus z_1 is common fixed point of A, B, S and T.

To prove uniqueness we suppose that p and q are two common fixed point of A, B, S and T such that p ≠ q, then from (3.1.1) we have,

$$M(Ap, Bq, z, qt) \geq \alpha_1 \min\begin{Bmatrix} M(Sp, Tq, z, t), \\ M(Sp, Ap, z, t), \\ M(Ap, Bq, z, t) \end{Bmatrix} + \alpha_2 \min\begin{Bmatrix} M(Bq, Tq, z, t), \\ M(Ap, Tq, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min \begin{Bmatrix} M(Bq, Sp, z, t), \\ M(Ap, Bq, z, t) \end{Bmatrix} + \alpha_4 M(Ap, Bq, z, t)$$

$$M(p, q, z, qt) \geq \alpha_1 \min \begin{Bmatrix} M(p, q, z, t), \\ M(p, p, z, t), \\ M(p, q, z, t) \end{Bmatrix} + \alpha_2 \min \begin{Bmatrix} M(q, q, z, t), \\ M(p, q, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min \begin{Bmatrix} M(q, p, z, t), \\ M(p, q, z, t) \end{Bmatrix} + \alpha_4 M(p, q, z, t)$$

$$M(p, q, z, qt) \geq \alpha_1 \min \begin{Bmatrix} M(p, q, z, t), \\ 1, \\ M(p, q, z, t) \end{Bmatrix} + \alpha_2 \min \begin{Bmatrix} 1, \\ M(p, q, z, t) \end{Bmatrix}$$

$$+ \alpha_3 \min \begin{Bmatrix} M(q, p, z, t), \\ M(p, q, z, t) \end{Bmatrix} + \alpha_4 M(p, q, z, t)$$

$$M(p, q, z, qt) \geq (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) M(p, q, z, t) > M(p, q, z, t)$$

$$M(p, q, z, kt) \geq M(p, q, z, t);$$

Lemma 2.7 implies that $p = q$. This completes the proof of the theorem.

Remark 3.2: In light of [3], we proved common fixed point theorem for common E.A. like property in fuzzy 2-metric space and generalized and improve the result of Subhani and Kumar [13].

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