GREEN'S FUZZY CONGRUENCE OF IDEMPOTENT SEPARATING FUZZY RELATION

Dr. G. Hariprakash

Principal

Jayasree Arts and Science College, Kalanadikolly, Pulpally-673579, Wynad (Dist), Kerala, India

Abstract

The object of the work discussed in this paper is about the classes of fuzzy Green's relations. The aim of this study is to characterize the 'transformation semigroups' and 'semigroups of transformations' using fuzzy property to get semigroups of fuzzy transformations and transformation fuzzy subsemigroups. In particular a subsemigroup of classes of fuzzy Green's relations is considered to introduce a semigroup of Green's fuzzy transformations. Any of its subgroup is defined as a fuzzy transformation subsemigroup. Based on these concepts six prepositions are established to find out a theorem stating equivalent conditions of strictly idempotent separating fuzzy congruence equivalence relations on a generalized inverse semigroup. From three theorems, state a Green's fuzzy idempotent separating fuzzy relation between two elements of a generalized inverse semigroup.
The study concludes by establishing this fuzzy relation as a fuzzy congruence.

As a result a new notion is introduced by defining the fuzzy relation between these two elements called Green's fuzzy congruence of idempotent separating relation.

Keywords : **Fuzzy Green's relations, Semigroup of fuzzy transformations, Fuzzy transformation subsemigroups. Green's fuzzy congruence of idempotent separating relations**.

BASIC CONCEPTS

Fuzzy relations have an important role in Mathematics. It indicates the strength of association between elements of n-tuple. If the fuzzy relation defined the strength of association between elements of 2-tuple it is called a fuzzy binary relation. A fuzzy relation which is reflexive, symmetric and transitive is called a similarity relation. A fuzzy compatible similarity relation (Clifford and Preston, 1961&1967) is called a fuzzy congruence. In this work $con_f(S)$ denotes the set of all fuzzy congruences (Kim, and Bae, 1997) on S. An element $\mu \in con_f(S)$ is said to be idempotent separating if $\mu_e = \mu_f \Rightarrow e = f$ and strictly idempotent separating if $\mu_e = \mu_f \Leftrightarrow e = f$ where $e, f \in E(s)$.

GREEN'S FUZZY RELATIONS (Green, 1951, Hariprakash, 2016)

Let μ is a fuzzy binary relation on *a* semigroup *S* the fuzzy Green $\hat{\mathcal{L}}$, $\hat{\mathcal{R}}$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{H}}$ -relations are defined as follows.

For any $a, b \in S$, $a\ \hat{\mathcal{L}}\ b$ or $(a, b) \in \hat{\mathcal{L}}$ if

$S'_a = S'_b$ and $\mu(a, x) = \mu(b, x)\ \forall \in S$

$$\hat{\mathcal{L}}(a, b) = \max_{z,w\in S} \min\{\mu(a, z), \mu(b,w)\}$$

$$\hat{\mathcal{L}} = \{(a, b) \in S \times S : S'_a = S'_b \text{ and } \mu(a, x) = \mu(b, x)\ \forall \in S\}$$

For $a, b \in S$; $a\hat{\mathcal{R}}\ b$ or $(a, b) \in \hat{\mathcal{R}}$ if

$aS' = bS'$ and $\mu(x, a) = \mu(x,b)\ \forall \in S$

$$\hat{\mathcal{R}}(a, b) = \max_{z,w\in S} \min\{\mu(z, a), \mu(w,b)\}$$

$$\hat{\mathcal{R}} = \{(a, b) \in S \times S : aS' = bS' \text{ and } \mu(x, a) = \mu(x, b)\ \forall \in S\}$$

$\hat{\mathcal{D}} = \hat{\mathcal{L}} \vee \hat{\mathcal{R}}$ and $\hat{\mathcal{H}} = \hat{\mathcal{L}} \wedge R$.

Definition 2.1.

Let S be a semigroup. The subsemigroup $T_{\hat{\mathcal{L}}}(S)$ of $\beta_{\hat{\mathcal{L}}}(S)$ of all fuzzy Green's $\hat{\mathcal{L}}$-relations, satisfying the following conditions is called semigroup of $\hat{\mathcal{L}}$ fuzzy transformations:

(i) For any $x \in S$ there exists $y \in S$ such that $(x,y) \in \hat{\mathcal{L}}$, where $\hat{\mathcal{L}} \in \beta_{\hat{\mathcal{L}}}(S)$

(ii) $(x,y),(x,y^1) \in \hat{\mathcal{L}} \Rightarrow y = y^1$

Similarly, the subsemigroup $T_{\hat{\mathcal{R}}}$ (s) of $\beta_{\hat{\mathcal{R}}}$ (s) of all fuzzy Green's $\hat{\mathcal{R}}$ relations satisfying the following conditions is Called semigroup of $\hat{\mathcal{R}}$ -fuzzy transformations:

(i) For any $x \in S$, there exists $y \in S$ such that $(x,y) \in \hat{\mathcal{R}}$.

(ii) $(x,y),(x,y^1) \in \hat{\mathcal{R}} \Rightarrow y = y^1$.

Definition 2.2.

Any subsemigroup of semigroup of $\hat{\mathcal{L}}$-fuzzy transformation $T_{\hat{\mathcal{L}}}$ (s) is called a $\hat{\mathcal{L}}$-fuzzy transforamtion semigroup. Similarly, any subsemigroup of semigroup of $\hat{\mathcal{R}}$ -fuzzy transformation $T_{\hat{\mathcal{R}}}$ (s) is called a $\hat{\mathcal{R}}$ -fuzzy transformation semigroup.

Proposition 2.3.

Let $T_{\hat{\mathcal{L}}}(s)$ be a semigroup of $\hat{\mathcal{L}}$-fuzzy transformations on S, where each $\hat{\mathcal{L}}$ is a fuzzy congruence Green's $\hat{\mathcal{L}}$-relation on S. Then $(a, b) \in T_{\hat{\mathcal{L}}}(s) \Rightarrow a = b$.

Proof.

Let $(a,b) \in T_{\hat{\mathcal{L}}}(s)$. Then $(a,b) \in \hat{\mathcal{L}}$ for some $\hat{\mathcal{L}} \in T_{\hat{\mathcal{L}}}(s)$.

Since $\hat{\mathcal{L}}$ is a fuzzy congruence, $(a, a) \in \hat{\mathcal{L}}$. That is, (a, a) $\in \hat{\mathcal{L}}$ and $(a, b) \in \hat{\mathcal{L}}$. Then, by definition 2.1, $a = b$.
Hence the result.

Proposition 2.4.

Let $T_{\hat{\mathcal{R}}}$ (s) be a semigroup of $\hat{\mathcal{R}}$ -fuzzy transformations on S, where each $\hat{\mathcal{R}}$ is a fuzzy congruence Green's $\hat{\mathcal{R}}$ -relation on S. Then

$$(a,b) \in T_{\hat{\mathcal{R}}}(s) \Rightarrow a = b.$$

Proof.

Result follows from proposition 2.3 using the properly of $\hat{\mathcal{R}}$ instead of $\hat{\mathcal{L}}$.

Proposition 2.5.

The semigroup of $\hat{\mathcal{L}}$-fuzzy transformations of a fuzzy congruence Green's $\hat{\mathcal{L}}$-relations on a semigroup S is a subset of $\hat{\mathcal{L}}^{-1}$(l).

Proof.

Let $(a,b) \in T_{\hat{\mathcal{L}}}(s)$. Then, by proposition 2.3, $a = b$. But,

$a = b \Rightarrow \hat{\mathcal{L}}_a = \hat{\mathcal{L}}_b \Rightarrow \hat{\mathcal{L}}(a, b) = 1 \Rightarrow (a, b)\hat{\mathcal{L}}^{-1}(1)a = b.$

Hence $T_{\hat{\mathcal{L}}}(s) \leq L^{-1}(1)$.

Proposition 2.6.

The semigroup of $\hat{\mathcal{R}}$ -fuzzy transformations of a fuzzy congruence Green's $\hat{\mathcal{R}}$ -relations on a semigroup S is a subset of $\hat{\mathcal{R}}$

Proof.

Result follows from proposition 2.5, using the property of $\hat{\mathcal{R}}$ instead of $\hat{\mathcal{L}}$.

Proposition 2.7.

If $\hat{\mathcal{L}}$ is a fuzzy congruence Green's $\hat{\mathcal{L}}$ relation on a semigroup S, its semigroup of $\hat{\mathcal{L}}$ -fuzzy transformations is a subset of $\hat{\mathcal{L}}$.

Proof.

Since $\hat{\mathcal{L}}$ is a fuzzy congruence relation on S, by proposition 2.5,

$$T_{\hat{\mathcal{L}}}(s) \leq \hat{\mathcal{L}}^{-1}(1). \qquad (1)$$

Also, $\hat{\mathcal{L}}^{-1}(1) \leq \hat{\mathcal{L}}$. Therefore, from (1) $T_{\hat{\mathcal{L}}}(s) \leq \hat{\mathcal{L}}$.

Proposition 2.8.

If $\hat{\mathcal{R}}$ is a fuzzy congruence Green's $\hat{\mathcal{R}}$ relation on S, its semigroup of $\hat{\mathcal{R}}$ -fuzzy transformations is a subset of $\hat{\mathcal{R}}$.

Proof.

Result follows from proposition 2.7 using the property of $\hat{\mathcal{R}}$.

Theorem 2.9.

Let S denote a generalized inverse semigroup (Madhavan, 1998, Kuroki, 1997)and $\hat{\mathcal{L}}$ a strictly idempotent separating fuzzy congruence Green's fuzzy relation on S. Then the following conditions are equivalent:

1. There exists an idempotent e in S, such that $\hat{\mathcal{L}}(a, be) = 1$.

2. For any $a' \in V(a)$, $\hat{\mathcal{L}}(a, ba'a) = 1$.

Proof.

Assume that there exists an idempotent e in S such that $\hat{\mathcal{L}}(a, be)= 1$. Since $\hat{\mathcal{L}}$ is a fuzzy congruence $\hat{\mathcal{L}}_a = \hat{\mathcal{L}}_{be}$.
Again, $\hat{\mathcal{L}}$ is strictly idempotent separating so $\hat{\mathcal{L}}_a = \hat{\mathcal{L}}_{be} \Leftrightarrow a = be$. Since a belongs to the generalized inverse semigroup S, there exists $a^* \in V(a)$, $b' \in V(b)$ such that a^* is an inverse of be. That is,
$a^* =(be)^{-1}= eb^1$.
We have,

$$\hat{\mathcal{L}}_{ba'a} = \hat{\mathcal{L}}_{beb'a} = \hat{\mathcal{L}}_{bb'beb'be}$$
$$= \hat{\mathcal{L}}_{bb'bb'bee}, \text{ since } S \text{ is a generalized inverse semigroup.}$$
$$= \hat{\mathcal{L}}_{bb'bee} = \hat{\mathcal{L}}_{be}$$
$$= \hat{\mathcal{L}}_{a} \qquad (2)$$

Now, for $a' \in V(a)$,

$$\hat{\mathcal{L}}_{ba'a} = \hat{\mathcal{L}}_{ba'aa'a} = \hat{\mathcal{L}}_{bb'ba'aa'aa'a} = \hat{\mathcal{L}}_{bb'ba'aa'aa'a} = L_{ba'aa'a}$$
$$= \hat{\mathcal{L}}_{ba'a} = \hat{\mathcal{L}}_{a}, \text{ from (2)}$$

That is $\hat{\mathcal{L}}_{ba'a} = \hat{\mathcal{L}}_a$ implies $\hat{\mathcal{L}}(ba'a,a)= 1$

Conversely, assume $\hat{\mathcal{L}}(ba'a,a)= 1$, then

$$\hat{\mathcal{L}}_{ba'a} = \hat{\mathcal{L}}_{a} = \hat{\mathcal{L}}_{b(a'a)}$$

Here $a'a$ an idempotent in S, hence (1) is obvious. That is, the given two conditions are equivalent.

Definition 2.10.

Let a and b be any two elements of a generalized inverse semigroup S. Define a fuzzy relation $\omega_{\hat{\mathcal{L}}}$ called a Green's $\hat{\mathcal{L}}$-fuzzy congruence of idempotent separating relation such that ${}_a\omega_{\hat{\mathcal{L}}}\, b$ if one of the conditions of theorem 2.9, and hence both the conditions are satisfied. Also, $\omega_{\hat{\mathcal{L}}}(a,b) = \hat{\mathcal{L}} \circ \hat{\mathcal{L}}(a,b)$

Theorem 2.11.

$\omega_{\hat{\mathcal{L}}}$ is a fuzzy congruence.

Proof.

We have $\omega_{\hat{\mathcal{L}}}(a,b) = \hat{\mathcal{L}} \circ \hat{\mathcal{L}}(a,b)$. Then

$$\omega_{\hat{\mathcal{L}}}(a,a) = \hat{\mathcal{L}} \circ \hat{\mathcal{L}}(a,a)$$
$$= \max_{z\in S} \min\{\hat{\mathcal{L}}(a,z), \hat{\mathcal{L}}(z,a)\}$$
$$\geq \min\{\hat{\mathcal{L}}(a,a), \hat{\mathcal{L}}(a,a)\}$$
$$\geq \min\{1,1\}$$
$$\geq 1.$$

Since $\omega_{\hat{\mathcal{L}}}$ is a fuzzy relation, $\omega_{\hat{\mathcal{L}}} \leq 1$. Hence, $\omega_{\hat{\mathcal{L}}}(a,a) = 1$. That is,
$\omega_{\hat{\mathcal{L}}}$ is reflexive.

$$\omega_{\hat{\mathcal{L}}}(a,b) = \hat{\mathcal{L}} \circ \hat{\mathcal{L}}(a,b)$$
$$= \max_{z\in S} \min\{\hat{\mathcal{L}}(a,z), \hat{\mathcal{L}}(z,b)\}$$
$$= \max_{z\in S} \min\{\hat{\mathcal{L}}(z,b), \hat{\mathcal{L}}(a,z)\}$$
$$= \max_{z\in S} \min\{\hat{\mathcal{L}}(b,z), \hat{\mathcal{L}}(z,a)\}$$

$= \hat{\mathcal{L}} \circ \hat{\mathcal{L}}(b,a)$
$= \omega_{\hat{\mathcal{L}}}(b,a)$

Hence, $\omega_{\hat{\mathcal{L}}}$ is symmetric.
Again, we have

$$\omega_{\hat{\mathcal{L}}} \circ \omega_{\hat{\mathcal{L}}}(a,b) = \max_{z\in S} \min\{\omega_{\hat{\mathcal{L}}}(a,z), \omega_{\hat{\mathcal{L}}}(z,b)\}$$
$$= \max_{z\in S} \min\{\hat{\mathcal{L}} \circ \hat{\mathcal{L}}(a,z), \hat{\mathcal{L}} \circ \hat{\mathcal{L}}(z,b)\}$$
$$\leq \max_{z\in S} \min\{\hat{\mathcal{L}}(a,z), \hat{\mathcal{L}}(z,b)\}$$
$$\leq \hat{\mathcal{L}} \circ \hat{\mathcal{L}}(a,b)$$
$$\leq \omega_{\hat{\mathcal{L}}}(a,b).$$

Hence $\omega_{\hat{\mathcal{L}}}$ is transitive.
Also, we have

$$\omega_{\hat{\mathcal{L}}}(a,b) = \hat{\mathcal{L}} \circ \hat{\mathcal{L}}(a,b)$$
$$= \max_{z\in S} \min\{\hat{\mathcal{L}}(a,z), \hat{\mathcal{L}}(z,b)\}$$
$$= \max_{z\in S} \min\{\hat{\mathcal{L}}(at,z), \hat{\mathcal{L}}(z,b)\}, t \in S$$
$$= \hat{\mathcal{L}} \circ \hat{\mathcal{L}}(at,b)$$
$$= \omega_{\hat{\mathcal{L}}}(at,b)$$

Similarly, we get

$$\omega_{\hat{\mathcal{L}}}(a,b) \leq \in \omega_{\hat{\mathcal{L}}}(a,bt)$$

Hence, $\omega_{\hat{\mathcal{L}}}$ is fuzzy right compatible. In the same way we get that $\omega_{\hat{\mathcal{L}}}$ is fuzzy left compatible. That is, $\omega_{\hat{\mathcal{L}}}$ is fuzzy compatible. Thus, $\omega_{\hat{\mathcal{L}}}$ is a fuzzy compatible similarity relation. That is, $\omega_{\hat{\mathcal{L}}}$ is a fuzzy congruence. Hence the result.

Theorem 2.12.

Let S denote a generalized inverse semigroup and $\hat{\mathcal{R}}$ strictly idempotent separating fuzzy congruence Green's relation on S. Then the following conditions are equivalent for any $a,b \in S$:

1. There exists an idempotent e in S, such that $\hat{\mathcal{R}}(a,eb) = 1$.
2. For any $a' \in V(a)$ $\hat{\mathcal{R}}(a,aa'b) = 1$.

Proof.

Result follows from theorem 2.9 using the property of $\hat{\mathcal{R}}$ .instead of $\hat{\mathcal{L}}$.

Definition 2.13.

Let a and b be any two elements of a generalized inverse semigroup S. Define a fuzzy relation $\omega_{\hat{\mathcal{R}}}$, called the Green's $\hat{\mathcal{R}}$ -fuzzy congruence of idempotent separating fuzzy relation such that $_a\omega_{\hat{\mathcal{R}}}\, b$ if and only if one of the condition of theorem 2.12, and hence, both the conditions are satisfied.

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