ON SCHUR M- POWER CONVEXITY FOR PROPORTION OF DISTINCTION OF SOME SPECIAL MEANS IN TWO VARIABLES

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Abstract. In this paper, we discuss the Schur m-power convexity on $(0,\infty)\times(0,\infty)$. For proportion of distinction of some special means in two variables, such as arithmetic, geometric, harmonic, root-square means and the like, and obtain some inequalities related to proportion of distinction of means.

1. MEAN OF ORDER t

Let us consider the following well known *mean of order t* :

(1.1) $$B_t(a,b) = \begin{cases} \left(\dfrac{a^t+b^t}{2}\right)^{1/t}, & t\neq 0 \\ \sqrt{ab}, & t=0 \\ \max\{a,b\}, & t=\infty \\ \min\{a,b\}, & t=-\infty \end{cases}$$

for all $a,b,t\in R,\ a,b>0.$
In particular, we have

$$B_{-1}(a,b)=H(a,b)=\frac{2ab}{a+b}$$
$$B_0(a,b)=G(a,b)=\sqrt{ab}$$
$$B_{1/2}(a,b)=N_1(a,b)=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2$$
$$B_1(a,b)=A(a,b)=\frac{a+b}{2}$$
$$B_2(a,b)=S(a,b)=\sqrt{\frac{a^2+b^2}{2}}$$

The means, *H (a, b), G(a, b), A(a, b) and S(a, b)* are known in the literature as *harmonic*, *geometric*, *arithmetic* and *root-square means* respectively. For simplicity we can call the measure, $N_1(a, b)$ as *square-root* mean. It is well know that [1] the *mean of order* s given in (1.1) is monotonically increasing in *s*, then we can write

(1.2) $$H(a,b)\le G(a,b)\le N_1(a,b)\le A(a,b)\le S(a,b)$$

Dragomir and Pearce [3] (page 242) proved the following inequality:

(1.3) $$\frac{a^r+b^r}{2} \le \frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)} \le \left(\frac{a+b}{2}\right)^r$$

Let us consider two parameter

$$\frac{a^r+b^r}{2} \le \frac{\frac{b^{r+1}-a^{r+1}}{(r+1)(b-a)}}{\frac{b^{s+1}-a^{s+1}}{(s+1)(b-a)}} \le \left(\frac{a+b}{2}\right)^r$$

(1.4) $$\frac{a^r+b^r}{2} \le \frac{(s+1)(b^{r+1}-a^{r+1})}{(r+1)(b^{s+1}-a^{s+1})} \le \left(\frac{a+b}{2}\right)^r$$

For all $a, b > 0$, $a, b > 0, a \neq b, r \in (0,1), and\ s \in (0,1)$. in particular take $r = \frac{1}{2}$ we get

$$\frac{\sqrt{a}+\sqrt{b}}{2} \le \frac{2(s+1)\left(b^{3/2}-a^{3/2}\right)}{3(b^{s+1}-a^{s+1})} \le \left(\frac{a+b}{2}\right)^{1/2}$$

Multiply by $\frac{\sqrt{a}+\sqrt{b}}{2}$

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \le \frac{(s+1)\left(b^{\frac{3}{2}}-a^{\frac{3}{2}}\right)(\sqrt{a}+\sqrt{b})}{3(b^{s+1}-a^{s+1})} \le \left(\frac{a+b}{2}\right)^{1/2}\frac{\sqrt{a}+\sqrt{b}}{2}$$

(1.5) $$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \le \frac{(s+1)(b-a)(a+b+\sqrt{ab})}{3(b^{s+1}-a^{s+1})} \le \left(\frac{a+b}{2}\right)^{1/2}\frac{\sqrt{a}+\sqrt{b}}{2}$$

If $s=0$

(1.6) $$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \le \frac{(a+b+\sqrt{ab})}{3} \le \left(\frac{a+b}{2}\right)^{1/2}\frac{\sqrt{a}+\sqrt{b}}{2}$$

i.e., $N_1(a,b) \le H_e(a,b) \le N_2(a,b)$

If $s=1/2$ from (1.5)

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \le \frac{(b-a)}{2(\sqrt{b}-\sqrt{a})} \le \left(\frac{a+b}{2}\right)^{1/2}\frac{\sqrt{a}+\sqrt{b}}{2}$$

(1.7) $$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \le \frac{\sqrt{a}+\sqrt{b}}{2} \le \left(\frac{a+b}{2}\right)^{1/2}\frac{\sqrt{a}+\sqrt{b}}{2}$$

If $s=1$ from (1.5)

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \le \frac{2(b-a)(a+b+\sqrt{ab})}{3(b^2-a^2)} \le \left(\frac{a+b}{2}\right)^{1/2}\frac{\sqrt{a}+\sqrt{b}}{2}$$

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 \le \frac{2(a+b+\sqrt{ab})}{3(a+b)} \le \left(\frac{a+b}{2}\right)^{1/2}\frac{\sqrt{a}+\sqrt{b}}{2}$$

On the other side we can easily check that

$$\left(\frac{a+b}{2}\right)^{1/2}\frac{\sqrt{a}+\sqrt{b}}{2}\leq\frac{a+b}{2}$$

$$\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^{2}\leq\frac{N_3}{A}\leq\left(\frac{a+b}{2}\right)^{1/2}\frac{\sqrt{a}+\sqrt{b}}{2}$$

Here $N_1=\left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2$, $N_2=\left(\frac{a+b}{2}\right)^{1/2}\frac{\sqrt{a}+\sqrt{b}}{2}$, $N_3=\frac{a+b+\sqrt{ab}}{3}$, and $N_4=\frac{N_3}{A}$

Finally the expressions (1.5), (1.6) lead us to the following inequalities [7]

$$H(a,b)\leq G(a,b)\leq N_1(a,b)\leq H_e(a,b)\leq N_2(a,b)\leq A(a,b)\leq S(a,b)$$

Let us consider the following distinction of means was studied in [17].

(1.8) $M_{SA}(a,b)=S(a,b)-A(a,b)$

(1.9) $M_{SN_2}(a,b)=S(a,b)-N_2(a,b)$

(1.10) $M_{SH_e}(a,b)=S(a,b)-H_e(a,b)$

(1.11) $M_{SN_1}(a,b)=S(a,b)-N_1(a,b)$

(1.12) $M_{SG}(a,b)=S(a,b)-G(a,b)$

(1.13) $M_{SH}(a,b)=S(a,b)-H(a,b)$

(1.14) $M_{AN_2}(a,b)=A(a,b)-N_2(a,b)$

(1.15) $M_{AG}(a,b)=A(a,b)-G(a,b)$

(1.16) $M_{AH}(a,b)=A(a,b)-H(a,b)$

(1.17) $M_{N_2N_1}(a,b)=A(a,b)-N_2(a,b)$

(1.18) $M_{N_2G}(a,b)=N_2(a,b)-G(a,b)$

Clearly the above distinction of means are nonnegative and convex in $\Re^2_+=(0,\infty)\times(0,\infty)$.
In this paper, we defined new distinction of means and by using these we define some special means, then we discussed "Schur m - power convexities for special means ".

2.Special Means

From the distinction of means defined in the equations 1.8 to 1.18, we define the following difference between the means

(2.1) $M_{SN_2}(a,b)-M_{SA}(a,b)=A-N_2$

(2.2) $M_{SH_e}(a,b)-M_{SN_2}(a,b)=N_2-H_e$

(2.3) $M_{SH_e}(a,b)-M_{SA}(a,b)=A-H_e$

(2.4) $M_{SN_1}(a,b)-M_{SH_e}(a,b)=H_e-N_1$

(2.5) $M_{SN_1}(a,b)-M_{SN_2}(a,b)=N_2-N_1$

(2.6) $M_{N_1H}(a,b)-M_{GH}(a,b)=N_1-G$

(2.7) $M_{SH}(a,b)-M_{SG}(a,b)=G-H$

Clearly, the above difference of the means are convex for all positive real value of `t'.
Now, by using the (2.1) to (2.7) difference of means, we establish the special means as follows:

$$(2.8) \qquad \frac{M_{SN_2} - M_{SA}}{M_{SH_e} - M_{SN_2}} = \frac{A - N_2}{N_2 - H_e}$$

$$(2.9) \qquad \frac{M_{SN_1} - M_{SH_e}}{M_{NH} - M_{GH}} = \frac{H_e - N_1}{N_1 - G}$$

Lemma 2.1. In [14] J.Rooin and M Hassni , introduced the homogeneous functions, the function

(2.10) $f(x) = \frac{a^x - b^x}{c^x - d^x}$ and $g(x) = \frac{a^x - b^x}{c^x - d^x}$. For $a > b \geq c > d > 0$ where $x \in (-\infty, \infty)$ is

(i) Convex, if $ad - bc > 0$

(ii) Concave, if $ad - bc < 0$ and

(iii) Equality holds, if $ad - bc = 0$

3. Preliminaries

We begin with recalling some basic concepts and notations in the theory of majorization.
For more details, we refer the reader to [1,2].

Definition 3.1 . Let $x = (x_1, x_2, x_3, ..., x_n)$ and $y = (y_1, y_2, y_3, ..., y_n) \in R^n$

i) x is said to be majorized by y (in symbols $x \prec y$), $\sum_{i=1}^{k} x_{[i]} \leq \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, 3..., n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$ where $x_{[1]} \geq ... \geq x_{[n]}$ and $y_{[1]} \geq ... \geq y_{[n]}$ are rearrangement of x and y in a descending order.

ii) $\Omega \subset R^n$ is called a convex set, if $(\alpha x_1 + \beta y_1, \alpha x_2 + \beta y_2, ..., \alpha x_n + \beta y_n) \in \Omega$, for any $\boldsymbol{x}$ and $y \in \Omega$, where α and $\beta \in [0,1]$ with $\alpha + \beta = 1$

iii) Let $\Omega \subset R^n$, the function $\varphi : \Omega \to R^n$ is said to be schur convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \leq \varphi(y)$. φ is said to be a Schur concave function on Ω, if and only if $-\varphi$ is Schur convex function.

Definition 3.2 . Let $x = (x_1, x_2, x_3, ..., x_n)$ and $y = (y_1, y_2, y_3, ..., y_n) \in R^n{}_+$.

$\Omega \subset R^n$ is called geometrically convex set, if $(x_1^{\alpha} y_1^{\beta}, x_2^{\alpha} y_2^{\beta}, ..., x_n^{\alpha} y_n^{\beta}) \in \Omega$, for any $\boldsymbol{x}$ and $y \in \Omega$, where $\alpha, \beta \in [0,1]$ with $\alpha + \beta = 1$.

A generalization of Schur convex functions was introduced by Yang [14], as follows

Definition 3.3. Let $f : R_{++} \to R$ be defined by

$$f(x) = \begin{cases} \dfrac{x^m - 1}{m} & , m \neq 0 \\ \ln\ r, & m = 0 \end{cases}$$

Lemma 2.4. Let $\varphi : \Omega \to R_+$ be continuous on Ω and differentiable on Ω^0. Then φ is Schur m-power convex on function $x = (x_1, x_2, x_3, ..., x_n) \in \Omega^0$ if and only if φ is symmetric on Ω and

$$\frac{{x_1}^m - {x_2}^m}{m}\left[{x_1}^{m-1}\frac{\partial\varphi(x)}{\partial x_1} - {x_2}^{m-1}\frac{\partial\varphi(x)}{\partial x_2}\right] \geq 0\ if\ m \neq 0$$

and

$$(\log x_1 - \log x_2)\left[{x_1}^m\frac{\partial\varphi(x)}{\partial x_1} - {x_2}^m\frac{\partial\varphi(x)}{\partial x_2}\right] \geq 0\ if\ m = 0$$

4. Main Results

In this paper, we discuss the Schur m-power Convexity of the distinguishes special means, in the following theorems.

Theorem 4.1. For m ≠ 0, the proportion of distinction of means $\dfrac{M_{SN_2} - M_{SA}}{M_{SH_e} - M_{SN_2}}$ is Schur m-power convex function in R_+^2.

Proof. Let

$$f(a,b) = \frac{M_{SN_2} - M_{SA}}{M_{SH_e} - M_{SN_2}} = \frac{A - N_2}{N_2 - H_e}$$

By Lemma 2.1

$$f(a,b) = AH_e - N_2^2$$

$$f(a,b) = \frac{a^2 + b^2 + 2ab - 2a\sqrt{ab} - 2b\sqrt{ab}}{24}$$

by finding the partial derivatives of $f(a,b)$ and simple manipulation gives we have

$$\frac{\partial f}{\partial a} = \frac{2a + 2b - 3\sqrt{ab} - b\left(\frac{\sqrt{b}}{\sqrt{a}}\right)}{24}$$

$$\frac{\partial f}{\partial b}=\frac{2b+2a-3\sqrt{ab}-a\left(\frac{\sqrt{a}}{\sqrt{b}}\right)}{24}$$

By m -power Schur convexity,

$$\Delta=\frac{a^m-b^m}{m}\left[a^{1-m}\frac{\partial f}{\partial a}-ab^{1-m}\frac{\partial f}{\partial b}\right]$$

$$=\frac{a^m-b^m}{24m}\left[\left(2a+2b-3\sqrt{ab}\right)\left(a^{1-m}-b^{1-m}\right)-\frac{ab\sqrt{b}}{a^m\sqrt{a}}+\frac{ab\sqrt{a}}{b^m\sqrt{b}}\right]$$

$$=\frac{a^m-b^m}{24ma^mb^m}\left[\left(2a+2b-3\sqrt{ab}\right)\left(ab^m-ba^m\right)-\sqrt{ab}\left(b^{m+1}-a^{m+1}\right)\right]$$

$$=\frac{a^m-b^m}{24ma^mb^m}\left[\left(2a+2b-3\sqrt{ab}\right)\left(ab^m-ba^m\right)-\sqrt{ab}\left(b^{m+1}-a^{m+1}\right)\right]$$

$$=\frac{a^m-b^m}{24ma^mb^m}\left[\left(2a+2b-3\sqrt{ab}\right)\left(ab^m-ba^m\right)+\sqrt{ab}\left(a^{m+1}-b^{m+1}\right)\right]$$

Thus $\Delta\geq 0$. From Lemma 3.2 ,it follows that the proportion of distinction of mean is Schur 'm'-power convex functions in R^n_{++} .

Theorem 4.2. For m $\neq 0$, the proportion of distinction of means $\dfrac{M_{SH_e}-M_{SA}}{M_{SN_1}-M_{SH_e}}$ is Schur m-power convex function in R^2_+.

Proof. Let

$$f(a,b)=\frac{M_{SH_e}-M_{SA}}{M_{SN_1}-M_{SH_e}}=\frac{A-H_e}{H_e-N_1}$$

By Lemma 3.2

$$f(a,b)=AN_1-H_e^2$$

$$f(a,b)=\frac{a^2+b^2 6ab-2a\sqrt{ab}+2b\sqrt{ab}}{72}$$

by finding the partial derivatives of $f(a,b)$ and simple manipulation gives we have

$$\frac{\partial f}{\partial a}=\frac{2a-6b+3\sqrt{ab}+b\left(\frac{\sqrt{b}}{\sqrt{a}}\right)}{72}$$

$$\frac{\partial f}{\partial b}=\frac{2b-6a+3\sqrt{ab}+a\left(\frac{\sqrt{a}}{\sqrt{b}}\right)}{72}$$

By m -power Schur convexity,

$$\Delta=\left(\frac{a^m-b^m}{m}\right)\left[a^{1-m}\frac{\partial f}{\partial a}-ab^{1-m}\frac{\partial f}{\partial b}\right]$$

$$=\left(\frac{a^m-b^m}{72m}\right)\left[\left(3\sqrt{ab}\right)\left(a^{1-m}-b^{1-m}\right)+a\left(2a^{1-m}+6b^{1-m}\right)-b\left(6a^{1-m}+2b^{1-m}\right)+\frac{ab\sqrt{b}}{a^m\sqrt{a}}-\frac{ab\sqrt{a}}{b^m\sqrt{b}}\right]$$

$$=\left(\frac{a^m-b^m}{72m}\right)\left[\frac{1}{a^m}\left(2a^2+\sqrt{ab}(3a+b)-6ab\right)+\frac{1}{b^m}\left(2b^2-\sqrt{ab}(3b+a)+6ab\right)\right]$$

$$=\left(\frac{a^m-b^m}{72ma^mb^m}\right)\left[b^m\left(2a^2+\sqrt{ab}(3a+b)\right)+a^m\left(2b^2-\sqrt{ab}(3b+a)\right)+6ab\left(a^m-b^m\right)\right]\geq 0$$

Thus $\Delta\geq 0$. From Lemma 3.2 ,it follows that the proportion of distinction of mean is Schur 'm'-power convex functions for $x\in R_{++}^n$.

5.Conclusion

In this paper, we distinguished special means and discussed about the Schur properties of Schur `m'-power convexities on the proportion of distinction of special means.

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