Fuzzy gt-set and fuzzy gt-nowhere dense sets.

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Abstract: In this paper we introduce a new concept of fuzzy set theory that is fuzzy gt-set, fuzzy gt-G_δ-set, fuzzy gt-F_σ-set, fuzzy gt-dense set, fuzzy gt-nowhere dense set. Several properties are also discussed. Illustrate with suitable examples.

Keywords: Fuzzy sets, fuzzy locally open sets, fuzzy locally closed sets, fuzzy gt-sets, fuzzy gt-dense sets, and fuzzy gt-nowhere dense sets.

1. Introduction

In order to deal with uncertainties, the idea of fuzzy sets and fuzzy set operations was introduced by L. A. Zadeh [8] in the year 1965. This inspired mathematician to fuzzify Mathematical Structures. The first notion of fuzzy topological space had been defined by C. L. Chang in 1968 [2]. The fuzzy nowhere dense set were introduced and studied by the authors in Dr. G. Thangaraj and Dr. G. Balasubramanian [7]. The fuzzy locally nowhere dense set were introduced and studied by the authors in Dr. S. Anjalmose and A. Saravanan [1]. In this paper we introduce a new class of fuzzy gt-sets, (in the name of Professor G. Thangaraj, simply gt) fuzzy gt-dense sets, fuzzy gt-nowhere dense sets. Several properties are also discussed with suitable examples.

2. Preliminaries

Definition 2.1: [3]

By a fuzzy topological space a non - empty set X together with a fuzzy topology T (in the sense of Chang) and denote it by (X, T).

Let λ and μ be any two fuzzy sets in (X, T). Then we define $\lambda \vee \mu : X \to [0,1]$ and $\lambda \wedge \mu : X \to [0,1]$ as follows: $(\lambda \vee \mu)(x) = \text{Max}\{\lambda(x), \mu(x)\}$ and $(\lambda \wedge \mu)(x) = \text{Min}\{\lambda(x), \mu(x)\}$.

Let (X, T) be any fuzzy topological space and λ be any fuzzy set in (X, T). We define $Cl(\lambda) = \wedge\{\mu / \lambda \leq \mu, 1-\mu \in T\}$ and $int(\lambda) = \vee\{\mu / \mu \leq \lambda, \mu \in T\}$. For any fuzzy set λ in a fuzzy topological space (X, T).

Definition 2.2: [5]

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy G_δ - set in (X, T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for $i \in I$.

Definition 2.3: [5]

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy F_σ - set in (X, T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$, where $\lambda_i \in T$ for $i \in I$.

Definition 2.4: [4]

A subset β of a fuzzy topological space X is called fuzzy locally closed set if $\beta = \alpha \wedge \delta$, where α is a fuzzy-open set and δ is fuzzy-closed set.

The complement of fuzzy-locally closed set is called fuzzy-locally open set.

Definition 2.5: [7]

A fuzzy set λ in a fuzzy Topological space (X; T) is called fuzzy dense if there exists no fuzzy closed set in (X; T) such that $\lambda < \mu < 1$.

Definition 2.6: [6]

A fuzzy set λ in a fuzzy topological space (X,T) is called fuzzy nowhere dense if there exists no non-zero fuzzy open set μ in (X,T) such that $\mu < cl(\lambda)$. That is, $int\ cl(\lambda) = 0$.

Definition 2.7: [1]

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy locally dense if there exists no fuzzy locally closed set μ in (X, T) such that $\lambda < \mu < 1$.

Definition 2.8: [1]

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy locally nowhere dense if there exists no non-zero fuzzy locally-open set μ in (X,T) such that $\mu < l\text{-}cl(\lambda)$. That is, $l\text{-}int\ \ l\text{-}cl(\lambda) = 0$.

3. Fuzzy gt-set.

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy gt-closed if $\lambda = \mu \wedge \gamma$, where μ is fuzzy closed and γ is fuzzy locally open in (X, T). The complement of fuzzy gt-closed is fuzzy gt-open.

Example 3.1:

Let X = {a, b}. The fuzzy sets λ, and μ are defined on X as follows:

$\lambda : X \to [0,1]$ defined as $\lambda(a) = 0.2$; $\lambda(b) = 0.7$, $\mu : X \to [0,1]$ defined as $\mu(a) = 0.1$; $\mu(b) = 0.5$, Then T = {0, λ, μ, 1} is a fuzzy topology on X.

The fuzzy sets $\lambda \wedge (1-\lambda) = \alpha$ (say), $\lambda \wedge (1-\mu) = \beta$ (say), $\mu \wedge (1-\lambda) = \eta$ (say), $\mu \wedge (1-\mu) = \zeta$ (say), therefore the fuzzy sets α, β, η, ζ are fuzzy locally closed sets.

Now the fuzzy sets $(1-\lambda) \wedge (1-\alpha) = 1-\lambda$, $(1-\mu) \wedge (1-\alpha) = 1-\beta$, $(1-\mu) \wedge (1-\eta) = 1-\mu$ are fuzzy gt-closed sets, where 1-λ, 1-μ are fuzzy closed and 1-α, 1-β, 1- η, 1- ζ are fuzzy locally open set in (X,T). The fuzzy sets λ, β, μ are fuzzy gt-open sets in (X, T).

Proposition 3.1:

If λ is a fuzzy gt-closed in a fuzzy topological space (X, T) then λ is fuzzy locally open set in (X, T). Converse need not be true.

Proof:

In example 3.1, 1-β is fuzzy gt-closed set in (X, T) and fuzzy locally open set in (X, T).

Converse need not be true. Consider the above example.

In example 3.1, the fuzzy set 1-α is fuzzy locally open but not of fuzzy gt-closed set in (X, T).

Proposition 3.2:

If λ is a fuzzy closed in a fuzzy topological space (X, T) then λ is fuzzy gt-closed set in (X, T). Converse need not be true.

Proof:

In above example 3.1, 1-λ and 1-μ are fuzzy closed set in (X, T) and fuzzy gt-closed set in (X, T).

Converse need not be true. Consider the above example.

In example 3.1, the fuzzy set 1-β is fuzzy gt-closed but not of fuzzy closed set in (X, T).

Proposition 3.3:

If λ is a fuzzy open in a fuzzy topological space (X, T) then λ is fuzzy gt-open set in (X, T). Converse need not be true.

Proof:

In above example 3.1, λ and μ are fuzzy open set in (X, T) and fuzzy gt-open set in (X, T).

Converse need not be true. Consider the above example.

In example 3.1, the fuzzy set β is fuzzy gt-open but not of fuzzy open set in (X, T).

4. Fuzzy gt-G_δ-set, Fuzzy gt-$F\sigma$-set.

A Fuzzy set λ in a Fuzzy topological space (X; T) is called a Fuzzy gt-G_δ-set in (X, T) if $\lambda = \wedge_{i=1}^{\infty}(\lambda_i)$ where λ_i 's are fuzzy gt-open sets for i ϵ I, Consider the following example.

Example 4.1

Let X = {a, b, c}. The fuzzy sets λ, μ, and γ are defined on X as follows:

λ : X →[0,1] defined as λ(a) = 0.2; λ(b) = 0.3, μ:X →[0,1] defined as μ(a) = 0.1; μ(b) = 0.4.

Then T = {0, λ, μ, λ∨μ, λ∧μ, 1} is a fuzzy topology on X.

Now the fuzzy sets λ∧(1-λ) = λ , λ∧(1-μ) = λ, λ∧(1- λ∧μ) = λ, λ∧(1- λ∨ μ) = λ, μ∧(1-λ) = μ, μ∧(1- μ) = μ, μ∧(1- λ∧μ) = μ, μ ∧(1- λ∨μ) = μ, (λ∧μ) ∧(1-λ) = λ∧μ, (λ∧μ) ∧(1- μ) = λ∧μ, (λ∧μ) ∧(1-λ∧μ) = λ∧μ, (λ∨μ)∧(1-λ) = λ∨μ, (λ∨μ)∧(1-μ) = λ∨μ, (λ∨μ)∧(1-λ∧μ) = λ∨μ, (λ∨μ)∧(1-λ∨μ) = λ∨μ,

Therefore the fuzzy sets λ, μ, λ∧μ, λ∨μ are fuzzy locally closed sets, then 1-λ, 1-μ, 1-λ∧μ, 1-λ∨μ are fuzzy locally open sets.

Now (1-λ)∧(1-λ) = 1-λ, (1-λ)∧(1-μ) = 1-λ∨μ, (1-λ)∧(1-λ∧μ) = 1-λ, (1-λ)∧(1-λ∨μ), (1-μ)∧(1-λ) = 1-λ∨μ, (1-μ)∧(1-μ) = 1-μ, (1-μ)∧(1-λ∧μ) = 1-μ, (1-μ)∧ (1- λ∨μ) = 1-λ∨μ, 1-(λ∧μ) ∧(1-λ) = 1-(λ∧μ)∧(1-μ) = 1-μ, 1-(λ∧μ)∧(1-λ∧μ) = 1-λ∧μ, 1-(λ∧μ)∧(1-λ∨μ) = 1-λ∨μ, 1-(λ∨μ)∧(1-λ) = 1-λ∨μ, 1-(λ∨μ)∧(1-μ) = 1-λ∨μ, 1-(λ∨μ)∧(1-λ∧μ) = 1-λ∨μ, 1-(λ∨μ)∧(1-

$\lambda\vee\mu) = 1-\lambda\vee\mu$. Therefore the fuzzy sets 1-λ, 1-μ, 1-λ∧μ, 1-λ∨μ are fuzzy gt-closed set in (X, T), Hence $[\lambda\wedge\mu\wedge(\lambda\wedge\mu)\wedge(\lambda\vee\mu)] = \lambda\wedge\mu$ is fuzzy gt-G_δ-set in (X,T).

Proposition 4.1

A Fuzzy set λ in a Fuzzy topological space (X, T) is called a Fuzzy gt-F_σ-set in (X; T) if ¸ $\lambda = V_{i=1}^{\infty}(\lambda_i)$, where λ_i 's are fuzzy gt-closed sets for i ϵ I, Consider the following example.

Proof:

In the above example 4.1, the fuzzy sets 1-λ, 1-μ, 1-λ∧μ, 1-λ∨μ are fuzzy locally closed sets, then $[(1-\lambda)\vee(1-\mu)\vee(1-\lambda\wedge\mu)\vee(1-\lambda\vee\mu)] = 1-\lambda\wedge\mu$ is fuzzy gt-F_σ-set in (X, T).

5. Fuzzy gt-dense sets.

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy gt-dense if there exists no fuzzy gt-closed set μ in (X, T) such that $\lambda < \mu < 1$.

Example 3.1:

Let X = {a, b}. The fuzzy sets λ, and μ are defined on X as follows:

λ : X →[0,1] defined as λ(a) = 0.2; λ(b) = 0.7, μ:X →[0,1] defined as μ(a) = 0.1; μ(b) = 0.5, Then T = {0, λ, μ, 1} is a fuzzy topology on X.

The fuzzy sets λ∧(1-λ) = α (say), λ∧(1-μ) = β (say), μ∧(1-λ) = η (say), μ∧(1- μ) = ζ (say), therefore the fuzzy sets α, β, η, ζ are fuzzy locally closed sets.

Now the fuzzy sets (1-λ)∧ (1-α) = 1-λ, (1-μ)∧ (1-α) = 1-β, (1-μ)∧ (1- η) = 1-μ are fuzzy gt-closed sets, where 1-λ, 1-μ are fuzzy closed and 1-α, 1-β, 1- η, 1- ζ are fuzzy locally open set in (X,T). The fuzzy sets λ, β, μ are fuzzy gt-open sets in (X, T). The fuzzy set λ if fuzzy gt-dense in (X, T), since gt-cl(λ) = 1.

Proposition 3.6:

If a fuzzy locally dense set in a fuzzy topological space (X, T) is need not be fuzzy gt-dense set in (X, T). Consider the example.

Example 3.2:

Let X = {a, b, c}. The fuzzy sets λ, μ, and β are defined on X as follows:

λ : X →[0,1] defined as λ(a) = 0; λ(b) = 0.2; λ(c) = 0.5, μ:X →[0,1] defined as μ(a) = 0.1; μ(b) = 0.2; μ(c) = 0.7, β : X →[0,1] defined as β(a) = 0.3; β(b) = 0.5; β(c) = 0.9.

Then T = {0, λ, μ, β, 1} is a fuzzy topology on X.

Now the fuzzy sets λ∧(1-λ) = λ , λ∧(1-μ) = γ (say), λ∧(1-β) = δ (say), μ∧(1-λ) =η(say), μ∧(1- μ)=ζ(say), μ∧(1-β) =χ(say), β∧(1- λ) = κ(say), β∧(1- μ) = ν(say), β∧(1- β) = ι(say), therefore the fuzzy sets λ, γ, δ, η, ζ, χ, κ, ν, ι are fuzzy locally closed sets. The fuzzy sets 1-λ, 1-γ, 1-δ, 1-η, 1-ζ, 1- χ, 1- κ, 1-ν, 1-ι are fuzzy locally open sets,

Now since (1-λ)∧ (1-λ) = 1- λ, (1-λ)∧ (1- γ) = 1- λ, (1-λ)∧ (1- δ) = 1- λ, (1-λ)∧ (1- η) = 1- η, (1-λ)∧ (1- ζ) = 1- η, (1-λ)∧ (1- χ) = 1- η, (1-λ)∧ (1- κ) = 1- ι, (1-λ)∧ (1- ν) = 1- ι, (1-λ)∧ (1- ι) = 1- ι, (1-μ)∧ (1- λ) = 1- μ, (1-μ)∧ (1- γ) = 1- μ, (1-μ)∧ (1- δ) = 1- μ, (1-μ)∧ (1- η) = 1- μ, (1-μ)∧ (1- ζ) = 1- μ, (1-μ)∧ (1- χ) = 1- μ, (1-μ)∧ (1- κ) = α (say), (1-μ)∧ (1- ν) = α, (1-μ)∧ (1- ι) = α, (1-β)∧ (1-λ) = 1- β, (1-β)∧ (1- γ) = 1- β, (1-β)∧ (1- δ) = 1- β, (1-β)∧ (1- η) = 1- β, (1-β)∧ (1- ζ) = 1- β, (1-β)∧ (1- χ) = 1- β, (1-β)∧ (1- κ) = 1- β, (1-β)∧ (1- ν) = 1- β, (1-β)∧ (1- ι) = 1- β. Therefore the fuzzy sets1- λ, 1- η, 1- ι, 1- μ, α, 1- β, are fuzzy gt-closed, and then the fuzzy sets λ, η, ι, μ, 1- α, β are fuzzy gt-open. The fuzzy set 1- η is fuzzy locally dense but not of fuzzy gt-dense in (X, T).

6. Fuzzy gt-nowhere dense sets.

A fuzzy set λ in a fuzzy topological space (X, T) is called fuzzy gt-nowhere dense if there exists no non-zero fuzzy gt-open set μ in (X,T) such that μ < gt-cl(λ). That is, gt-int gt-cl(λ) = 0.

Example 5.1:

Let X = {a, b}. The fuzzy sets λ, and μ are defined on X as follows:

λ : X →[0,1] defined as λ(a) = 0.2; λ(b) = 0.7, μ:X →[0,1] defined as μ(a) = 0.1; μ(b) = 0.5, Then T = {0, λ, μ, 1} is a fuzzy topology on X.

The fuzzy sets λ∧(1-λ) = α (say), λ∧(1-μ) = β (say), μ∧(1-λ) = η (say), μ∧(1- μ) = ζ (say), therefore the fuzzy sets α, β, η, ζ are fuzzy locally closed sets.

Now the fuzzy sets (1-λ)∧ (1-α) = 1-λ, (1-μ)∧ (1-α) = 1-β, (1-μ)∧ (1- η) = 1-μ are fuzzy gt-closed sets, where 1-λ, 1-μ are fuzzy closed and 1-α, 1-β, 1- η, 1- ζ are fuzzy locally open set in (X,T). The fuzzy sets λ, β, μ are fuzzy gt-open sets in (X, T). The fuzzy set 1- λ, is fuzzy gt-nowhere dense set, since gt-int gt-cl(1-λ) = 0. 1-β and 1-μ are not of fuzzy gt-nowhere dense sets, since gt-int gt-cl(1-β) ≠ 0 and gt-int gt-cl(1-μ) ≠ 0.

Proposition 5.1:

If λ is fuzzy gt-nowhere dense set in (X, T), then gt-int(λ) = 0

Proof:

Let λ is fuzzy gt-nowhere dense set in (X, T), therefore gt-int gt-cl(λ) = 0. Now λ ≤ gt-cl(λ) implies that gt-int (λ) ≤ gt-int gt-cl(λ). Hence gt-int (λ) = 0.

Proposition 5.2:

If λ is a fuzzy gt-nowhere dense set in (X, T), then (1-λ) is fuzzy gt-dense set in (X, T).

Proof:

If λ is a fuzzy gt-nowhere dense set in (X, T), By proposition 5.1, gt-int(λ) = 0. Now gt-cl(1-λ) = 1- gt-int(λ) = 1 – 0 = 1. Hence 1- λ is fuzzy gt-dense in (X, T).

Proposition 5.3:

If λ and μ are fuzzy gt-nowhere dense sets in a fuzzy topological space (X, T), then λ∧μ is also a fuzzy gt-nowhere dense set in (X, T).

Proof:

Let λ and μ is a Fuzzy gt-nowhere dense set in a fuzzy topological space (X; T). Then by proposition 5.1, gt-int(λ) = 0 and gt-int(μ)=0. Now gt-int(λ∧μ)= gt-int(λ) ∧ gt-int(μ) = 0 ∧ 0 = 0. Hence λ∧μ is fuzzy gt-nowhere dense set in (X, T).

Proposition 5.4

If λ ≤ μ and μ is a fuzzy gt-nowhere dense set in a fuzzy topological space (X, T), then λ is also fuzzy gt-nowhere dense set in (X, T).

Proof:

Now λ ≤ μ implies that gt-int gt-cl(λ) ≤ gt-int gt-cl(μ). Since μ is a fuzzy gt-nowhere dense set, hence gt-int gt-cl(μ) = 0. Then gt-int gt-cl(λ) = 0. Hence λ is a fuzzy gt-nowhere dense set in (X, T).

Proposition 5.5

If λ is a fuzzy gt-nowhere dense set and μ is any fuzzy set in a fuzzy topological space (X, T), then (λ^μ) is a fuzzy gt-nowhere dense set in (X, T).

Proof:

Let λ be a fuzzy gt-nowhere dense set in (X,T), then gt-int gt-cl(λ) = 0. Now (λ^μ) = gt-int gt-cl(λ^μ) = gt-int gt-cl(λ) ∧ gt-int gt-cl(μ) = 0 ∧ μ = 0. Hence (λ^μ) is fuzzy gt-nowhere dense set in (X, T).

Proposition 5.6:

If λ is a fuzzy gt-nowhere dense set and fuzzy gt-closed in a fuzzy topological space (X, T), then 1 – gt-cl(λ) is a fuzzy gt-dense set in (X, T).

Proof

Let λ be a fuzzy gt-nowhere dense set in (X, T). Now gt-cl(λ) = λ, since λ is fuzzy gt-closed set. Therefore 1-gt-cl(λ) = 1- λ. Hence by proposition 5.2, 1- λ is fuzzy gt-dense set in (X, T).

Proposition 5.7:

If λ is a fuzzy gt-closed set in a fuzzy topological space (X; T) with gt-int(λ) = 0, then λ is a fuzzy gt-nowhere dense set in (X; T).

Proof

Let λ be a fuzzy gt-closed and gt-int(λ) = 0, then gt-cl(λ) = λ, therefore gt-int gt-cl(λ) = gt-int(λ) = 0. Hence λ be a fuzzy gt-nowhere dense set in (X, T).

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