RECURRNCE RELATIONSIHIPS ON GENERATING FUNCTION

Vinit Kumar

Mathematics.

ABSTRACT :-

A detailed study by Godfrey Harold Hardy ($1877-1947$) and Srinivasa Ramanujan($1887-1920$) showed $P(j) \approx \frac{1}{4\sqrt{3}j} e^{\pi}\sqrt{2/3}\sqrt{j}$

This approximation was made exact by Hans Rademacher($1892-1969$) , who found an expansion that, when rounded to the nearest integer, gives $P(j)$ The next natural question is actually easier than the previous one, and is also due to De Movie.

INTRODUCTION-:

The number of partitions of j into exactly r parts where order counts is

$$\binom{j-1}{r-1}$$

Example

The partitions of 7 into four parts are

4+1+1+1 3+2+1+1 $2+3+1+1$ 1+3+2+1 2+2+2+1

1 +4+1+1 $3+1+2+1$ $2+1+3+1$ $1+3+1+2$ $2+2+1+2$

$1+1+4+1$ 3+1+1+2 $2+1+1+3$ 1+2+3+1 $2+1+2+2$ $1+1+1+4$

1+1+3+2 $1+1+2+3$

There are $20=\binom{4}{3}$ of them.

Proof 1

To each partition $a_1+a_2+\cdots+a_r$ we associate the sequence of partial sums

$$s_1 = a_1$$

$$s_2 = a_1 + a_2$$

$$s_3 = a_1 + a_2 + a_3$$

$$s_r = a_1 + a_2 + \cdots + a_r = j$$

Every partition gives rise to one and only one set of partial sums. Also every set of s's

$$0 < s_1 < s_2 < \cdots < s_{r-1} < j$$

determines one and only one partition. The number of partitions where order counts is therefore the number of ways of choosing the $(r-1)s$'s from the numbers $\{1,2,\cdots,j-1\}$, which is $\binom{j-1}{r-1}$.

Proof 2

Let us restate this problem as one of putting j identical balls in r different boxes, so that no box remains empty. If we represent this situation by slashes and t's we then intefpret the condition that no boxes remain empty as meaning that each slash (except for the end ones) must be between two $t's$. The r boxes are demarked by $r+1$ slashes two of which are external. The $(r-1)$ internal slashes can be placed in any subset of the $(j-1)$ spaces between the $jt's$. The number of ways of doing this is $\binom{j-1}{r-1}$.

Example

$|t|tt|t|ttt|$denotes six balls in four boxes. One in the first box, two in the second box, one in the third, and three in the last. It corresponds to

$$j = 7 = 1 + 2 + 1 + 3$$

Proof 3

The generating function for the number of partitions of j into exactly r parts is

$$G(t) = (t + t^2 + t^3 + \cdots)^r$$

since we have eliminated the option of choosing $1 = t^0$ from each term.

$$(t + t^2 + t^3 + \cdots)^r = t^r(1 + t^2 + t^3 + \cdots)^r$$

$$= t^r[\binom{r-1}{0} + \binom{r}{1}t + \binom{r+1}{1}t^2 + \cdots]$$

The coefficient of t^j in this product is just the coefficient of t^{j-r} in the bracketed series. This is

$$\binom{r-1+(j-r)}{(j-r)} = \binom{j-1}{j-1} = \binom{j-1}{r-1}$$

we will demonstrate that sometimes a recurrence relationship alone, without the recourse to generating functions, can be used to solve problems. We will consider a problem of Pierre Rémond de Montmort ($1678 - 1719$) , called "*le problem des rencontres*"

Let $(a_1, a_2, \ldots, a_j)$ be a permutation of the numbers $1,2,\cdots,j$ such that no element is back in its original place, that is, $a_1 \neq 1, a_2 \neq 2, a_j \neq j$. Such a

permutation is called a derangement. Let D_j be the number of derangements of the set $\{1,2,\cdots,j\}$.

Example

If we start with the 24 permutations of the numbers 1,2,3,4 and cross off all those with one in the first place, two in the second place, three in the third place, or four in the fourth place we have left these nine arrangements:

2143 2341 2413

3142 3412 3421

4123 4312 4321

Therefore, $D_4 = 9$.

Let $D_{13} = 1$and $D_1 = 0$. Let us distinguish two kinds of derangements. We know that a_1 sits in the first position; suppose that I sits in the a_1 th position, that is, a_1 and 1 just changed places. The rest of the $(j-2)$ numbers must form a smaller derangement with each element moved from its initial posi tion. This can happen in D_{j-2} ways. Since a_1 itself can be chosen in $(j-1)$ways the number of derangements of this kind is $(j-1)D_{j-2}$. We can now count the number of derangements in which 1 is not in the a_1 th position. First we can choose a_1 in $(j-1)$ ways. Now add it to the front of any derangement of $\{2,3,\ldots,j\}$ in which we have replaced the a_1 by 1. Since a_1 was not in place a_1 , I will not be in place a_1. This process will produce all the derangements of the second kind. Clearly there are $(j-1)D_{j-1}$ of these Adding both together we find,

$$D_j = (j-1)D_{j-1} + (j-1)D_{j-2}$$

Let us write this as,

$$\frac{D_j}{j!} = \frac{j-1}{j}\frac{D_{j-1}}{(j-1)!} + \frac{j-1}{j(j-1)}\frac{D_{j-2}}{(j-2)!}$$

We now introduce the notation

$$E_j = \frac{D_j}{j!} \quad E_0 = 1 \quad E_1 = 0$$

The E's satisfy the recurrence relationship

$$E_j = (1 - \frac{1}{j})E_{j-1} + \frac{1}{j}E_{j-2}$$

Thís can be rewritten in the form,

$$E_j - E_{j-1} = (\frac{-1}{j})(E_{j-1} - E_{j-1})$$

Reiterating this equation for $(j - 1)$ instead of (n) we obtain the descent,

$$= \frac{1}{j}\left(\frac{-1}{j-1}\right)(E_{j-2} - E_{j-3})$$

$$= \left(\frac{-1}{j}\right)\left(\frac{-1}{j-1}\right)\left(\frac{-1}{j-2}\right)(E_{j-3} - E_{j-t})$$

$$= \frac{(-1)^j}{j!}$$

We write this now in the form

$$E_j = \frac{(-1)^j}{j!} + E_{j-1}$$

Reiterating this equation for $(j - 1)$ instead of (n) gives us another descent,

$$= \frac{(-1)^j}{j!} + \frac{(-1)^{j-1}}{(j-1)!} + E_{j-2}$$

$$= (-1)^j[\frac{1}{j!} - \frac{1}{(j-1)!} + \frac{1}{(j-2)!} - + \cdots]$$

And so we have proven,

Theorem 30

$$D_j = j!\ (1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots \pm \frac{1}{j!})$$

Recall that

$$e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + - \cdots$$

This means that $D_,$, and $j!/e$ differ by less than $1/(j + 1)$ and so D_j is the closest integer to $j!/e. For$ any j the probability that any given permutation is a derangement is very close to $1/e$.

REFRENCES

- G.E.Andrews, A Generalization of the Collnitz-Gordon Partition Theorems, Proc,Amer.Math.Soc., 18, (1967), 945-952.

- G.E.Andrews, A Generalization of the Classical Partition Theorems, Trans.Amer_Math.Soc., 145, (1969), 205-221.
- G. E. Andrews , Some new partition theorems, J. Combinatorial Theory (to appear).
- W. N. Bailey, Some identities in combinatorial analysis, Proc. London Math.Soc. (2) 49 (1947), 421-435.
- Cohen-basic-techniques-of-combinatorial-theory