A NOTE ON REGULAR SEQUENCE SPACE

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Abstract : K***ö***the and Toeplitz, (1) have carried out works on sequence spaces some of which were later on developed by Allen, (1) and Dienes,P., (1). an account of all these can be found out in [Cooke, (1),chapter 10]. In this direction of study efforts were made to establish some of the results. we experienced that in sequence spaces αβ-limit implies c-limit of the same sequence in a sequence space. But there we did not get an authentic signal for a c-limit to be αβ-limit under αβ-convergence. In order to examine the problems of existence and the position of these limits Dienes,(1) introduced the notion of regular sequence spaces. Our sincere efforts in this paper is to study sequence spaces in detail to observe that under what circumstances some of the sequence spaces fall to be regular.

Keywords: Linear Space, Sequence Space, Dual Sequence Space, Perfect Sequence Space, Normal Sequence Space, Regular Sequence Space, Co-Ordinate Limit,Projective limit.

I. INTRODUCTION

As my abstract of this paper says that during to examine the problems of existence and the position of these limits Dienes,(1) introduced the notion of regular sequence spaces. Our sincere efforts in this paper is to study sequence spaces to observe that under what circumstances. some of the sequence spaces fall to be regular. Therefore our this paper of this little work will be wholly dedicated to establish some of the results on the formation of regular sequence spaces.

II. RESEARCH METHODOLOGY

Mainly analytical, fundamental, conceptual and qualitative research methodologies are adopted in this research paper. Which can be explained as-

2.1 Analytical research uses facts or information already available to analyse a critical evaluation of the research topic

2.2 Fundamental research is chief by concerned with formation of a theory and its generalization gathering knowledge for knowledge sake. i.e. in general we can say that research concerning some natural phenomena or relating to pure mathematics are all example of fundamental research.

2.3 Conceptual research is related to abstract ideas or theory which are not physical, It is generally used by thinkers and philosopher to put forth and develop new concepts or interpret existing theories in new way. Researcher in pure mathematics and natural sciences pursue this type of research in general.

2.4 This research is not quantitative research because we know that quantitative research is based on the measurement of some characteristics in terms of numerals. i.e.this research is done by qualitative research methodology too.

These are the why we used the above research methodologies.

III. DEFINITIONS

In this section we give some of the essential definitions by making the use of which we shall establish some of the results in the next section. It will be also to serve as a ready reference.

3.1 SEQUENCE : Let X be a non empty set then by a sequence in X we understand a mapping $f:N \rightarrow X$ of N into X, where N is the set of all natural numbers.Here the image of $n\epsilon N$ under f is denoted by f(n) or simply by f_n and we say it the n^{th} term of the sequence thus obtained.

Also such a sequence f is generally denoted by $\{f_n:n\epsilon N\}$ or $\{f_n\}$ or $<f_n:n\epsilon N>$ or $<f_n>$ or $f_1,f_2,f_3,\ldots\ldots\ldots,f_n,\ldots\ldots..$

Also a sequence is said to be a finite or an infinite sequence according as it contains finite or infinite number of terms respectively.

3.2 REAL SEQUENCE : When X=R (the set of all real numbers) then a sequence in X is called a real sequence or a sequence of real numbers. Evidently in this case for such a sequence f, every f_n is a real number.

<u>Examples of some of the sequences.</u>

Example 1. $a_n=\frac{n}{n+1}$ that is $<a_n> = ½,2/3,3/4,\ldots\ldots\ldots\ldots$

Example 2. $b_n=\frac{(-1)^n}{2^n}$ that is $<b_n> = -1/2,1/4,-1/8,1/16,\ldots$

Example 3 $a_n= n$ where $n\epsilon N$ then $<a_n> = 1,2,3,\ldots\ldots..$ is a sequence of natural number.

Example 4 $<a_n> = 2,3,5,7,11,13,\ldots\ldots\ldots.$ is a sequence of prime numbers.

3.3 POSITIVE SEQUENCE : A sequence $<f_n>$ in X is said to be positive if for some positive ϵ in X and some $K\epsilon N$, $f_n \geq \epsilon$ in X for all $n \geq K$ in N.

Example : The sequence <1/n> is not positive where as the sequence $<a+{}^{1}/{}_{n}>$ is positive for every a>0.

3.4 BOUNDED SEQUENCE : A sequence $\{f_n\}$ is said to be bounded if its range $\{f_n : n\epsilon N \}$ is a bounded set.That is when $m\leq f_n \leq M$ where m and M are the lower and upper bounds.Also it is worth much to note that $\{f_n\}$ or $\{ f_n : n\epsilon N\}$ denotes a sequence and is a function.But $\{f_n : n\epsilon N \}$ denotes the range of the sequence and is a set.

3.5 SUBSEQUENCE : Let $f_1,f_2,f_3,\ldots\ldots.,f_n,\ldots\ldots$ be a sequence. Now if $k_1,k_2,k_3,\ldots\ldots..$ is a sequence of natural numbers such that $k_1<k_2<k_3< \ldots\ldots\ldots..$ then a sequence of the form of $f_{k_1}, f_{k_2}, f_{k_3}, f_{k_4},\ldots\ldots\ldots$ is called a subsequence of the sequence $f_1,f_2,f_3,\ldots..$.

3.6 LIMIT OF A SEQUENCE : Let $\{x_n\}$ be a given sequence. Let l be a given number. Then l is said to be the limit of the sequence $\{x_n\}$ if for a given number ϵ, however small, we can find a positive integer m such that $|x_n - l| < \epsilon$ for all values $n \geq m$ and in this case we write $\lim_{n\to\infty} x_n = l$ or equivalently $n\to\infty$ implies that $x_n \to l$

Result 1: The limit of a sequence when it exists is unique.For the proof we refer to Natanson, (1)

3.7 CONVERGENCE OF A SEQUENCE : Let (f_n) be a sequence of points of R (the set of all real numbers) then (f_n) is said to converge to the real number l or equivelently the real number l is the limit of the sequence (f_n) if given any real number $\epsilon > o$, however small , there esists a natural number m such that

$$| f_n - l | < \epsilon \text{ for all } n \geq m$$

Here for the fact that l is the limit of the sequence $\{f_n\}$ may be denoted by writing $\lim_{n\to\infty} f_n = l$ or simply $\lim f_n = l$

or $f_n \to l$ as $n\to\infty$ or simply $f_n \to l$.

3.8 CONVERGENT SEQUENCE : A sequence which converges to a number l is said to be a convergent sequence.

Result 2: A convergent sequence has a unique limit.

Result 3: Every convergent sequence is bounded. For proof of the above two result we refer to Natanson (1).

3.9 LINEAR SPACE : A structure of linear space on a set E is defined by the two maps :

(a) $(x,y) \to x + y$ of $E \times E$ into E and is said to be vector addition .

(b) $(a,x) \to ax$ of $K \times E$ into E and is said to be scalar multiplication.

The above two maps are assumed to satisfy the following conditions :

(i) $x + y = y + x$ for every x,y in E .

(ii) $x + (y + z) = (x + y) + z$ for every x,y,z in E .

(iii) There exists an element 0 in E such that $x + 0 = 0 + x = x$ for every x in E.

(iv) For every element x in E there exists an element denoted by – x in E, such that $x + (- x) = (-x) + x = 0$ for every x in E.

(v) $a (x + y) = ax + ay$ for every a in K and all x,y in E.

(vi) $(a + b) x = ax + bx$ for every a,b in K and all x in E .

(vii) $(ab) x = a (bx)$ for every a,b in K and all x in E.

(viii) $1.x = x$ for every x in E.

Whenever all the above axioms are satisfied, we say that E is a linear space (or a vector space) over field K.

Now if K be the set of all real numbers then E is called a real linear space and similarly if K stands for the set of All complex numbers then E is called a complex linear space .Here every element of E is called a vector and every element of K is called a scalar. The zero vector 0 is unique and called the zero element or the origin in E.

3.10 SEQUENCE SPACE :A linear space whose elements are sequences is called a sequence space .

Thus a set E of sequences is a sequence space if , it contains the origin and for every x,y in E and for every scalar α , $x + y$ and αx are in E .

Definitions of some special sequence spaces are being given below making the use of which some comparative results have been established in this paper.

σ : The space of all sequences

S_0 : The zero sequence space which contains the origin only.

$\phi(s)$: The space of all finite sequences.

σ_∞ : The space of all bounded sequences.

$\Upsilon(s)$: The space of all convergent sequences.

Z : The space of all null sequence, in which $x_k \to 0$ as $k\to\infty$.

0_1 : The space of all sequence such that $x_{2k+1}=0$ for every k.

E_r : The space of all sequences such that $|x_k| < A^r (r>0)$ for every k

F_r : The space of all sequences such that $\sum_{k=1}^{\infty} k^r |x_k|$ converges (r>0).

σ_r : if $r \geq 1$, and if $\sum |x_k|^r$ and $\sum |y_k|^r$ converges then we have Minkowski's inequality

$$\left(\sum_{k=1}^{\infty} |x_k + y_k|^r\right)^{1/r} \leq \left(\sum_{k=1}^{\infty} |x_k|^r\right)^{1/r} + \left(\sum_{k=1}^{\infty} |y_k|^r\right)^{1/r}$$

Clearly the set of all $\{x_k\}$ such that $\sum |x_k|^r$ converges form a space denoted by σ_r.

Also the special case r=2 is Hilbert vector space σ_2.

σ_1 : σ_1 is the space of all $\{x_k\}$ such that $\sum |x_k|$ converges.

C : The space of all stationary sequences, in which $x_{k+1} = x_k$ for $k \geq k_0$.

δ :The space of all sequences such that if d_n is the number of non zero coordinates in the first n coordinates then

$$\lim_{n\to\infty} d_n \cdot n = 0$$

3.11 DUAL SPACE OF A SEQUENCE SPACE α (α^*) : Thus the dual space α^* of α is the set of points which can be projected on every direction in α.

Also α^* is a sequence space. Also α^* means the dual space of the sequence space α only.

The dual space of α^* is α^{**}, and evidently $\alpha^{**} \geq \alpha$.

Also we write $\alpha=\beta$ when the space α ,β are identical ,and $\alpha>\beta$ when α contains all points of β and some other point or points.

3.12 PERFECT SEQUENCE SPACE : A sequence space α is said to be prefect when $\alpha^{**}=\alpha$. [we refer to Cooke ,(1)]
Thus a perfect space contains every sequence which can be projected upon every direction in its dual space.
Clearly every prefect sequence space contains ϕ.

3.13 NORMAL SEQUENCE SPACE : A sequence space α is said to be normal if, whenever x is in α and $|y_k|\leq|x_k|$ for every k, then y is in α.

3.14 PROJECTIVE CONVERGENT (p-cgt in α relaive to β simply p-cgt or αβ- convergent) : If $\phi(s)\leq\beta\leq\alpha^*$ and if for sequence $x^{(n)}$in α , the sequence $u_n' = \sum_{k=1}^{\infty} x_k^{(n)} u_k$ converges for every u in β, we say that $x^{(n)}$ is projective convergent (p-convergent) relative to β or αβ- convergent.
When $\beta=\alpha^*$ (the dual space of α), we say that $x^{(n)}$ is projective convergent in α or α-convergent.

3.15 CO-ORDINATE LIMIT (or c-limit): If the $\lim_{n\to\infty} x_k^{(n)}$ exists for every k and is x_k then the point x= x_k is called the co-ordinate limit of $x^{(n)}$ and in this case we write c-lim $x^{(n)}$=x
3.16 PROJECTIVE LIMIT (p=limit) : A sequence x in α or outside α is called the projective limit (p-limit) of $x^{(n)}$ in α relative to β or αβ-limit $x^{(n)}$, when

(i) $\sum_{k=1}^{\infty} u_k x_k$ is absolutely convergent for every u_k in β.
That is x is in β^*, and

(ii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} x_k^{(n)} u_k = \sum_{k=1}^{\infty} x_k u_k$
for every u in β

When $\beta=\alpha_*$, x is called the projective limit of $x^{(n)}$ in α or α-lim $x^{(n)}$.
Also we refer to Cook, (1) to show that if αβ-limit $x^{(n)}$=x then c-limit $x^{(n)}$=x
Also it follows from(i) that c-limits of αβ-convergent sequences are considered as possible αβ-limits only if they are in β^*
Also by Cooke ,(1) αβ-convergence implies coordinate convergence (c-convergence) but the converse is false.

Result 4 : A necessary and sufficient condition for the αβ-convergence of $x^{(n)}$ in α is that to every u in β, and to every ϵ>0, there corresponds a positive number N (ϵ,u), such that for every $p,\varepsilon\geq N$,

$$|\sum_{k=1}^{\infty} u_k (x_k^{(p)} - x_k^{(\varepsilon)})|\leq\epsilon$$

Result 5: When β is normal, the necessary and sufficient condition that$x^{(n)}$ in α should be αβ-convergent is that to every u in β, and to every ϵ>0, corresponds a number N(ϵ,u) such that for every $p,\varepsilon\geq N$

$$\sum_{k=1}^{\infty} |u_k (x_k^{(p)} - x_k^{(\varepsilon)})|\leq\epsilon$$

3.17 REGULAR SEQUENCE SPACE :If with a given definition of convergence and limit , the c-limit of every convergent sequence in α is the limit of that sequence, then α is regular under the given convergence.

IV. RESULTS AND DISCUSSION

In this section we establish the following results by making use of the definitions given in section III of this paper.

THEOREM (1.4,I): Every perfect sequence space is normal.
PROOF: Let us suppose that the sequence space α is perfect.
Also let x is in α and that $|y_k|\leq|x_k|$ for every k.
Then $\sum |u_k x_k|$ converges.
For every $u\epsilon\alpha^*$
But $|y_k|\leq|x_k|$, so it is true that,
$\sum |u_k y_k|$ also converges
But then as u is in α^* so $y\epsilon\alpha^{**}$
But by hypothesis α is perfect
Thus $\alpha^{**}=\alpha$
Thus $y\epsilon\alpha^{**}$ implies that y is in α
Thus we have that for x is in α and $|y_k|\leq|x_k|$for every k y is in α.
Therefore α is normal.

THEOREM(1.4,II): For every sequence space α,α^* is normal.Where α^* is dual sequence space of sequence space α.
PROOF : Let $x\epsilon\alpha^*$ and $|y_k|\leq|x_k|$ for every k.
Thus $\sum_k |u_k x_k|$ converges
For $u\epsilon\alpha^{**}$
But then $\sum |u_k y_k|$ converges is true,
For $y\epsilon\alpha^*$
That is for $x\epsilon\alpha^*$ and $|y_k|\leq|x_k|$ for every k
Wefind that $y\epsilon\alpha^*$
Therefore α^* is normal.

THEOREM(1.4,III): A sequence space α is regular under αz-convergent.Where z is the space of null sequences in which $x_k\to 0$ as $k\to\infty$
PROOF : Since z is the space of null sequences in which $x_k\to 0$ as $k\to\infty$ that is $|x_k|<\epsilon$ for k is sufficiently large.
Now let $x\epsilon z$ and $|y_k|\leq|x_k|$ for every k.
Thus $|y_k|\leq|x_k|\leq\epsilon$ for every k

Thus $|y_k|\leq\epsilon$

But then y϶z

Hence z is Normal.

Let $x^{(n)}$ be a sequence of points in α.

Also let c-lim$x^{(n)}$=x ……………………….(1.11)

Now by Cooke,((1) theorem (10.2,I) (ii),

The necessary and sufficient condition that $x^{(n)}$ in α should be αβ-coinvergent is that to every u in β, and to everyϵ>0, corresponds a number N(ϵ,n) such that forevery p,$\mathcal{E}\geq$N, $\sum_{k=1}^{\infty}|u_k(x_k^{(p)}-x_k^{(\mathcal{E})}|\leq\epsilon$

Thus given any u in z and ϵ>0,

We have, $\sum_{k=1}^{\infty}|u_k(x_k^{(p)}-x_k^{(\mathcal{E})}|\leq\epsilon$

For p,$\epsilon\geq$N(ϵ,u)

Thus for every m, and for p,≥N

$\sum_{k=1}^{m}|u_k(x_k^{(p)}-x_k^{(\mathcal{E})}|\leq\epsilon$

If $\mathcal{E}$ is fixed and p increased, it follows from c-convergence

That $\sum_{k=1}^{m}|u_k(x_k-x_k^{(\mathcal{E})}|\leq\epsilon$, ……………….(1.12)

For $\mathcal{E}\geq$N and every m.

Letting m→∞, we have for $\mathcal{E}\geq$N,

$\sum_{k=1}^{\infty}|u_k(x_k-x_k^{(\mathcal{E})}|\leq\epsilon$, …………………………(1.13)

From (4.12) , we obtain

$\sum_{k=1}^{m}|u_kx_k|\leq\epsilon+\sum_{k=1}^{m}|u_kx_k^{(\mathcal{E})}|$,

But since $x^{(\mathcal{E})}$ is in α and u is in z≤α*, we have

[since by the definition of αβ-convergence α*⊇β]

$\lim_{m\to\infty}\sum_{k=1}^{\infty}|u_kx_k|$ converges, so that x is in z* (as u is in z)

Also by (1.13)

$\sum_{k=1}^{\infty}u_k(x_k-x_k^{(\mathcal{E})}|\leq\sum_{k=1}^{\infty}|u_k(x_k-x_k^{(\mathcal{E})}|\leq\epsilon$, For $\mathcal{E}\geq$N

Therefore $\lim_{\mathcal{E}\to\infty}\sum_{k=1}^{\infty}u_kx_k^{(\mathcal{E})}=\sum_{k=1}^{\infty}u_k\,x_k$

Thus αz-limit $x^{(n)}$=x ……………..(1.14)

Thus by [(1.11) and (1.14)] c-limit $x^{(n)}$= αβ-limit $x^{(n)}$ =x, 21thus α is regular under αz-convergence.

This result is the first outcomes of this paper that the sequence space α is regular under αz-convergence.

THEOREM (1.4,IV): A sequence space α is regular under ασ_1-convergence. Where σ_1 is the space of all $\{x_k\}$ such that $\sum|x_k|$ converges.

PROOF: Let x϶σ_1 and $|y_k|\leq|x_k|$

Then $\sum_k|x_ku_k|$ converges for u ϵ $\sigma_1^*=\sigma_\infty$

That $\sum_k|x_ku_k|$ converges

For u϶σ_∞

Hence $\sum_k|y_ku_k|$ will converge

For yϵ σ_∞^* as u is in σ_∞

But then y϶σ_1

Thus we get that

If x϶σ_1 and $|y_k|\leq|x_k|$ then y϶σ_1.

Thus σ_1 is normal.

Thus our theorem can be restated as . the sequence space α is regular under ασ_1-convergence where σ_1 is normal.

So we have that $x^{(n)}$ is necessary and sufficient condition for ασ_1-convergent and σ_1 is normal thus by the necessary and sufficient condition for αβ-convergent refer to section 1.2, given any u in σ_1 and any ϵ>0 we have

$\sum_k|u_k(x_k^{(p)}-x_k^{(\mathcal{E})})|\leq\epsilon$ For p,$\mathcal{E}\geq$N, $\sum_{k=1}^{m}|u_k(x_k^{(p)}-x_k^{(\mathcal{E})})|\leq\epsilon$.

Also αβ-convergent of $x^{(n)}$ implies c-convergent of $xd^{(n)}$

That is $\lim_{n\to\infty}x_k^{(n)}$ exists for every k. and in this case we write c-lim$x^{(n)}$=x.

Thus if $\mathcal{E}$ is fixed and p increased then it follows from c-convergences that

$\sum_{k=1}^{m}|u_k(x_k-x_k^{(\mathcal{E})})|\leq\epsilon$ ………………(1.15)

For $\mathcal{E}\geq$N and every m.

Now let m→∞ then we have for $\mathcal{E}\geq$N,

$\sum_{k=1}^{\infty}|u_k(x_k-x_k^{(\mathcal{E})}|\leq\epsilon$ ……………………………(1.16)

Now from (4.15), we get

$\sum_{k=1}^{m}|u_kx_k|\leq\epsilon+\sum_{k=1}^{m}|u_kx_k^{(\mathcal{E})}|$

$X^{(\mathcal{E})}$ is in α, and u is in $\sigma^1\leq\alpha^*$, we have

$\lim_{m\to\infty}\sum_{k=1}^{m}|u_kx_k^{(\mathcal{E})}|=\sum_{k=1}^{\infty}|u_kx_k^{(\mathcal{E})}|$,

Thus $\sum_{k=1}^{\infty}|u_kx_k|$ converges, so that x is in σ_1^*.

Also by (1.16),

$|\sum_{k=1}^{\infty}u_k(x_k-x_k^{(\mathcal{E})})|\leq\sum_{k=1}^{\infty}|u_k(x_k-x_k^{(\mathcal{E})}|\leq\epsilon$

For $\mathcal{E}\geq$N.

Thus $\lim_{\mathcal{E}\to\infty}\sum_{k=1}^{\infty}u_kx_k^{(\mathcal{E})}=\sum_{k=1}^{\infty}u_kx_k$

Hence αβ-lim$x^{(n)}$=x

But c-lim of αβ-convergent sequence $x^{(n)}$=x
Hence c-lim of $\alpha\sigma_1$-convergent sequence $x^{(n)}$ =αβ-lim$x^{(n)}$ =x
Thus α is regular.

THEOREM (1.4,V): The sequence space α is regular under $\alpha\Upsilon^*(s)$ convergence.Where ϒ(s):The space of all convergent sequences.
PROOF : Evidently $\Upsilon^*(s)$ is normal in the light of the theorem (1.4,II)
Let $x^{(n)}$ be a sequence of points in α.
Also let $x^{(n)}$ in α be $\alpha\Upsilon^*(s)$-convergent.
But αβ-convergent of $x^{(n)}$ implies c-convergent of $x^{(n)}$
Also c-convergent $x^{(n)}$ implies the existence of

$$\lim_{n\to\infty} x_k^{(n)} \text{ for every k.}$$

But then if this limit is x_k, the points x, constructed with coordinates x_k, is called the c-limit of $x^{(n)}$ and then we write
c-limit $x^{(n)}$ =x ……………………..(1.17)
we have to show that α is regular under $\alpha\Upsilon^*(s)$-convergence
for which it is sufficient to show that
c-limit of $x^{(n)}$=$\alpha\Upsilon^*(s)$-limit $x^{(n)}$
for this, since $x^{(n)}$ is $\alpha\Upsilon^*(s)$-convergent and $\Upsilon^*(s)$ is normal.
Hence in this case by a necessary and sufficient condition for αβ-convergence we have for an arbitrary ϵ>0 and u in $\Upsilon^*(s)$

$$\sum_{k=1}^{\infty} |u_k(x_k^{(p)} - x_k^{(\mathcal{E})}| \le \epsilon \text{ ………………….(1.18)}$$

For p,≥N(ϵ,u).
Thus for every m and for p,≥N.

$$\sum_{k=1}^{m} |u_k(x_k^{(p)} - x_k^{(\mathcal{E})}| \le \epsilon$$

If $\boldsymbol{\mathcal{E}}$ is fixed and p increased then it follows from c-convergence
That $\sum_{k=1}^{m} |u_k(x_k - x_k^{(\mathcal{E})}| \le \epsilon$ …………………(1.19)
For $\boldsymbol{\mathcal{E}}$≥N and every m.
Now assuming m→∞, we have for $\boldsymbol{\mathcal{E}}$≥N.

$$\sum_{k=1}^{\infty} |u_k(x_k - x_k^{(\mathcal{E})}| \le \epsilon \text{ ………………………..(1.20)}$$

But from (1.19)

$$\sum_{k=1}^{m} |u_k x_k| \le \epsilon + \sum_{k=1}^{m} |u_k x_k^{(\mathcal{E})}|$$

But $x^{(\mathcal{E})}$ϵα and uϵ$\Upsilon^*(s)$≤α^*
That is $x^{(\mathcal{E})}$ is in α and u is in α^* (the dual space of α)
Thus have that

$$\lim_{m\to\infty} \sum_{k=1}^{m} |u_k x_k^{(\mathcal{E})}| = \sum_{k=1}^{\infty} |u_k x_k^{(\mathcal{E})}$$

Hence $\sum_{k=1}^{\infty} |u_k x_k|$ converges, so that x is in $\Upsilon^{**}(s)$ …..(1.21)
And u is in $\Upsilon^*(s)$
Also from (1.20),

$$|\sum_{k=1}^{\infty} |u_k(x_k - x_k^{(\mathcal{E})}| \le \sum_{k=1}^{\infty} |u_k(x_k - x_k^{(\mathcal{E})}| \le \epsilon$$

For $\boldsymbol{\mathcal{E}}$≥N.

Thus $\lim_{\mathcal{E}\to\infty} \sum_{k=1}^{\infty} u_k x_k^{(\mathcal{E})} = \sum_{k=1}^{\infty} u_k x_k$ ………………..(1.22)
For every u in $\Upsilon^*(s)$
Thus due to the joint effort of (1.21) (1.22)
$\alpha\Upsilon^*(s)$-limit of $x^{(n)}$=x
but by (1.17)
c-limit $x^{(n)}$=x
thus c-limit of $\alpha\Upsilon^*(s)$-convergent sequence $x^{(n)}$=$\alpha\Upsilon^{*(s)}$-limit$x^{(n)}$=x
there for α is regular.

THEOREM (1.4,VI): Sequence space σ_∞ is regular under $\sigma_\infty\,\sigma_\infty^*$-convergence.
PROOF : Let xϵ σ_∞ and $|y_k|\le|x_k|$ for every k.
Then $\sum_k |x_k u_k|$ converges for uϵ σ_∞^*
Also $\sum_k |y_k u_k|$ will converges for yϵ σ_∞^{**}
But by privious result, σ_∞ is perfect
Thus $\sigma_\infty^{**} = \sigma_\infty$
Hence yϵ σ_∞^{**}⇒yϵ σ_∞
Therefore we find that
If xϵ σ_∞ and $|y_k|\le|x_k|$ for every k then yϵ σ_∞
thus σ_∞ is normal.
But by Theorem(1.3,II), σ_∞^* is also normal.
Now let $x^{(n)}$ be a sequence of points in σ_∞.
Also let $x^{(n)}$ in σ_∞ be $\sigma_\infty\,\sigma_\infty^*$-convergent.
That is $x^{(n)}$ in σ_∞ is σ_∞-convergent.
Also αβ-convergent of $x^{(n)}$ implies c-convergent of $x^{(n)}$
Thus $x^{(n)}$ is c-convergent
Then, $\lim_{n\to\infty} x_k^{(n)}$ exists for every k.

Then by definition if this limit is x_k is the c-limit of $x^{(n)}$ and then

c-limit $x^{(n)}=x$ ……………………..(1.23)

we have to show that σ_∞ is regular under σ_∞-convergent.

Which is possivle only when

c-limit of $x^{(n)}= \sigma_\infty$-limit of $x^{(n)}=x$

thus we have the situation that

$x^{(n)}$ is σ_∞-convergent and σ_∞^* is normal.

Thus by necessary and sufficient condition for αβ-convergence we can have for any $\epsilon>0$ and u in σ_∞^*,

$$\sum_{k=1}^{\infty}\left|u_k\left(x_k^{(p)}-x_k^{(\mathcal{E})}\right)\right| \le \epsilon \quad \text{………………………(1.24)}$$

For $p,\ge N(\epsilon,u)$.

Thus for every m, and for $p,\mathcal{E}\ge N$

$$\sum_{k=1}^{m}\left|u_k\left(x_k^{(p)}-x_k^{(\mathcal{E})}\right)\right| \le \epsilon$$

It $\mathcal{E}$ is fixed and p increased then it follows from c-convergence that

$$\sum_{k=1}^{m}\left|u_k\left(x_k-x_k^{(\mathcal{E})}\right)\right| \le \epsilon \quad \text{…………………(1.25)}$$

For $\mathcal{E}\ge N$ and every m.

Now letting $m\to\infty$, we have for $\mathcal{E}\ge N$,

$$\sum_{k=1}^{\infty}\left|u_k\left(u_k-x_k^{(\mathcal{E})}\right)\right| \le \epsilon \quad \text{…………………………(1.26)}$$

But from (4.25),

$$\sum_{k=1}^{m}|u_k x_k| \le \epsilon + \sum_{k=1}^{m}\left|u_k\left(x_k^{(\mathcal{E})}\right)\right|$$

But $x^{(\mathcal{E})}\epsilon\sigma_\infty$ and $u\epsilon\sigma_\infty^*$

Thus we have,

$$\lim_{m\to\infty}\sum_{k=1}^{m}\left|u_k x_k^{(\mathcal{E})}\right| = \sum_{k=1}^{\infty}\left|u_k x_k^{(\mathcal{E})}\right|$$

But $\sum_{k=1}^{\infty}|u_k x_k|$ will converse …………………(1.27)

For x is in $\sigma_\infty^{**}=\sigma_\infty$ and u is in σ_∞^*

Also from (1.26)

$$\left|\sum_{k=1}^{\infty}u_k\left(x_k-x_k^{(\mathcal{E})}\right)\right| \le \sum_{k=1}^{\infty}\left|u_k\left(x_k-x_k^{(\mathcal{E})}\right)\right| \le \epsilon$$

For $\mathcal{E}\ge N$

That is $\lim_{\mathcal{E}\to\infty}\sum_{k=1}^{\infty}u_k x_k^{(\mathcal{E})} = \sum_{k=1}^{\infty}u_k x_k$ ………………..(1.28)

For every u in σ_∞^*

Hence by (1.27) and (1.28)

$\sigma_\infty\sigma_\infty^*$-limit of $x^{(n)}=x$

but by (1.23)

x is the c-lim$x^{(n)}$

hence c-limit $x^{(n)}=\sigma_\infty\sigma_\infty^*$-limit of $x^{(n)}=x$.

hence σ_∞ is regular.

In addition to the results establishes above we can construct a few more sequence spaces in order to investigate .

Whether they are regular. But we give here a stoppage to the study of regular sequence spaces leaving the scope for others to study it further by establishing some other results.

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