CORE SOLUTION CONCEPT OF FUZZY GAMES

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ABSTRACT :

The aim of this paper is to apply fuzzy set theory to introduce fuzzy coalition and to obtain a set of allocations which can not be improved upon by fuzzy coalition.

1. INTRODUCTION :

There are so many categoreies of solution of games. One of the simplest solution concept of a game is the core. It was studied by Butnariu[6]. The core is the set of allocations which can not be improved upon by any coalition. Such set of allocations which can not be improved upon by fuzzy coalition has been introduced in this paper by defining fuzzy game without side payments.

2. FUZZY COALITIONS :

Let G = (N,v) be a nonfuzzy game of the set N = {1,2,3,……,n} of n players in which $v : S \rightarrow R$ is a real valued function (characterisitic function) from a family of coalition $S \subset N$ to the set of real numbers R. Hence v(A) means the gain which a coalition. A can aquire only through the action of A. The coalition A can be specified by the charactereistic function τ^A as follows :

$$\tau^A(i) \begin{cases} 1 \; if \; i \in A; \\ 0 \; if \; i \notin A. \end{cases}$$

A rate of participation τ^A (i) of a player i is defined by

τ^A (i) = 1, if a player i participates in A and

τ^A (i) = 0, if a player i does not participate in A.

Conseqently, a coalition A is represented by

$\tau^A = (\tau^A(1), \tau^A(2), \ldots\ldots\ldots\ldots\ldots\ldots \tau^A(n))$

A fuzzy coalition T is defined as a coalition in which a player i can participate with a rate of participation $\tau_i \in [0,1]$ instead of {o,1}. The characteristic function or coalitional worth function of a fuzzy game is a real valued function $f : [0,1]^n \rightarrow R$ which specifies a real number f(τ) of any fuzzy coalition τ,

This fuzzy game is denoted by FG = (N,f).

3. CORE OF FUZZY GAMES WITH SIDE PAYMENTS :

We define a "fuzzy game with side payments" by its "coalitional worth" function $f : [0,1]^n \to R$. such that $\tau \in [0,1]^n$ and $f'(0) = 0$

Here we assume the coalitional worth function to be positively homogeneous i.e. $f'(\lambda\tau) = \lambda f'(\tau)$, for all $\lambda \geq 0$.

Now we set up the coalitional worth function $f \to R^n +$ by

$$f'(\tau) = \left(\sum_{i \in N} \tau_i\right)^{f'} \left\{\frac{\tau}{\sum_{i \in N} \tau_i}\right\}$$

We say that a multiutility $c=(c_1, c_2, c_3 \ldots\ldots\ldots, c_n) \in R^n$ is improved upon by a fuzzy coalition τ if $\sum_{i \in N} \tau_i \; c_i < f'(\tau)$

The following statements are clearly equivalent :

(i) for all $\tau \sum_{i \in N} \tau_i \; c_i \geq f'(\tau)$

and $\sum_{i \in N} c_i = f'(\tau^N)$ ………….(A)

(ii) $c = \partial f'(\tau^N)$

where $\partial f'(\tau^N)$ denotes the differential of the function f' defined by

$$\partial f'(\tau) = \left\{c \in R^n \text{ such that } f'(\tau) - f'(\sigma) \geq \sum_{i \in N} c_i(\tau_i - \partial_i)\right\}$$

Now we define the "Core of the fuzzy game" by the set of multiutilities $c \in R^n$ satisfyng any one of (A).

Hence the following theorem holds :

4. THEOREM

If the coalitional worth function f of a fuzzy game with side payment is concave, then its core is convex, compact and non-empty. Furthermore, if f is differentiable at τ^N, then the core consists only of the gradient of f' at τ^N.

5. CORE OF FUZZY GAMES WITHOUT SIDE PAYMENT

We associate to a fuzzy coalition $T \in [0,1]$, the map.

$\tau^* : R^n \to R^n$ defined by

$$(\tau^* c)_i = \tau_i \; c_i \qquad \ldots\ldots\ldots\ldots[B]$$

and its support $A_\tau = \{i \in \frac{N}{\tau_i} > 0\}_n$, that is the subset of players i participating in the fuzzy coalition τ. We regard R^n as the "Space of multiutilities" and the sets $R^\tau = r^* R^n, R_+^{+\tau}, = \tau^* R_+^n, R_+^{+\tau} = \tau^* R_+^{+\tau}$ as the set of multiutilities of fuzzy coalition τ (respectively, positive, strongly positive multiutilities of τ)

6. DEFINTION

We define a "fuzzy game without side payments" f* by associating with any fuzzy coalition $\tau \in [Q,1]^n$ subsets of f*(τ) of multiutilites of fuzzy coalition τ satisfying the following conditions :

(i) f*(τ) is nonempty, closed and convex in $R^\tau = \tau^* R^n$;

(ii) f*(τ) is comprehensive (i.e. f*(τ)= f*(τ)- R_+^τ)
and bounded above (i.e. there exists C∈R, such that $f^*(\tau) \subset C - R_+^\tau$)

(iii) f* is positively homogeneous (i.e. for all $\lambda > 0, f^*(\lambda\tau) = \lambda f^*(\tau)$
This helps us to extend f* to R_+^n

Now we define the "Core of fuzzy game without side payments" to be the set of multiutilities satisfying the following conditions.

(i) $c \in f^*(\tau^N)$

(ii) for all $\tau \neq 0, \tau^* c \notin f^*(\tau) = f^*(\tau) - R_+^{+\tau}$

We say that a fuzzy coalition τ "improves upon" a multiutilities c if $\tau^* C \in f^*(\tau)$ i.e. if the fuzzy coalition τ can find a multiutility d ∈ P (τ) yielding to each player $i \in A_\tau$ a multiutility $d_i > c_i$. So, the core is the set of those multiutilities $c \in f^*(\tau^N)$ which can not be improved upon by any fuzzy coalition $\tau \neq 0$

Since the subsets $f^*(\tau)$ are closed and convex, so for any

$\lambda = (\lambda^1, \lambda^2, \ldots\ldots, \lambda^n) \in R^{\tau *}$, we associate f' (τ, λ) $\underset{C \in f^*(\tau)}{V} \; \sum_{i \in N} \lambda_i c_i$

Since f*(τ) is comprehensive and founded above, f (τ, λ) is finite iff $\lambda \in R_+^\tau$

The linear function λ on R^n are usually interpreted as "rate of transfer" of multiutilities, allowing players of pool a multiutlity $c=\{c_1, c_2, c_3 \ldots\ldots c_n\}$ to a common value utility $\sum_{i \in N} \lambda^i c_i$

7. THEOREM

Let c $\in f^*(\tau^N)$, and α be a function defined by

$$\alpha\ (c) = \underset{\tau\in[0,1]^n}{\vee}\ \underset{\lambda\in M^\tau}{\inf}[f'(\tau,\lambda) - \textstyle\sum_{i\in N}\lambda^i c_i)]$$

Then c belongs to the core iff $\alpha(c) \leq 0$.

Proof :

If c $\in f'(\tau^N)$ belongs to the core then $\tau^* c \notin f'(\tau), \forall\tau \in [0,1]^N$

Then there exists $\lambda \in R^{\tau^*}, \lambda \neq \phi$ such that $\sum_{i\in N}\lambda^i c_i \geq f'(\tau\lambda)$

Hence $\lambda \in R_+^\tau$ and we can take $\lambda \in M^\tau$, where

$M^\tau = \{\lambda \in R_+^{\tau^*}$ such that $\sum_{i\in A}\lambda^i = 1\}$ So, $\alpha(c) \leq 0$.

Conversely : If α (c) ≤ 0, then for any $\tau \in [0,1]^n$, there exists $\lambda \in M^\tau$ such that

$$f'(\tau,\lambda) - \sum_{i\in A_\tau}\lambda^i\tau_i c_i \leq \alpha(c)$$

As M^τ is compact and $\lambda \to f'(\tau\lambda)$, so, $\tau^* c \notin f^*(\tau)$

8. DEFINITION

We say $\underset{\lambda\in M^\tau}{\wedge}\left[f'(\tau,\lambda - \sum_{i\in N}\lambda^i\tau_i c_i\right]$ as the "complaint" of the fuzzy coalition τ and the function α, as the "maximum complaint function".

We define the "least core" of the game to be the subset of those c $\in f^*(\tau^N)$ that inimize the maximum complaint function α of f* (τ^N). The following theorem can easily be proved .

9. Theorem

The least core is not empty and contained in the subset of "Pareto-Optimal multiutilities". It is contained in the core whenever the latter is non-empty.

10. DEFINITION

Let $M^n = \left\{\lambda \in R^n+, \text{such that} \sum_{i\in N}\lambda^{i^n} = 1\right\}$

We say that a multiutility $\bar{c} \in f^*(t^N)$ is a "week canonical co-operative equilibrium" if there exists a rate of transfer

$$\bar{\lambda} \in \frac{M^n}{\forall\tau} \in [0,1]^n, \sum_{i \in N} \bar{\lambda}^i \tau_i c_i \geq f'(\tau\bar{\lambda}) \qquad \text{--------[C]}$$

We say that $\bar{C}$ is a 'strong canonical co-operative equilibrium if [C] holds.

11. THEOREM

(a) Any strong canonical co-operative equilibrium is contained in the core.

(b) If the game satisfies $f^*(\tau) + f^*(s) \subset f^*(\tau + \sigma), \forall \tau, \sigma \subset: R^N +$ ....[D]

then the core of the fuzzy game f^* is contained in the subset of weak canonical cooperative equilibria.

Proof :

(a) If $c \in f^*(\tau^N)$ is a strong equilibrium, then α (c) < 0 and so, c belong to the core by theorem 7.

(b) Letc $\subset$ $f^*(\tau^N)$ belongs to the core. Hence by theorem 7 we obtain

$$\bigvee_{\tau \in [0,1]^n} \bigwedge_{\lambda \in M^n} \left[f^*(\tau, \lambda) - \sum_{i \in N} \bar{\lambda} \tau_i c_i \right] \leq \alpha(c) \leq 0 \qquad \text{......(E)}$$

Assumption [D] implies that $t \to f'(\tau, \lambda)$ is concave. Since M^n is

convex and compact and $\lambda \to f'^{(\tau,\lambda)}_{i \in N} \sum \lambda^i \tau_i c_i$ is convex, so, there exists

$$\bar{\lambda} \in M^n \text{ such that } \bigvee_{\tau \in [0,1]^n} \left[f'(\tau, \bar{\lambda}) - \sum_{i \in N} \lambda^i \tau_i c_i \right]$$

$$= \bigvee_{\tau \in [0,1]^n} \bigwedge_{\lambda \in M^n} \left[f'(\tau, \bar{\lambda}) - \sum_{i \in N} \lambda^i \tau_i c_i \right] \qquad \text{--------------[F]}$$

From [E] and [F], the theorem holds.

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