DOMINATION AND IT'S TYPE IN GRAPH THEORY

Dr. Shobha Shukla[1]

Vikas Singh Thakur[2]

1. Associate Professor (Supervisor) Dept. of Mathematics

Dr. C.V. Raman University Kargi Road Kota, Bilaspur (C.G.), India.

2. Ph.D Scholar in Mathematics

Dr. C.V. Raman University Kargi Road Kota, Bilaspur (C.G.), India,

Abstract:

Dominating set in a graph G is a set of vertices D such that each vertex is either in D or has a neighbour in D. A partition of V such that each class is a dominating set in G is called a domatic partition of G.In this paper we first show some definitions & known results in the field, presenting fundamentals as well as more recent concepts in domination. In particular, we turn our attention to ordinary domination, factor domination (where D dominates every given spanning subgraph of G), and distance domination (where a vertex not in D is within a given distance from D).

Key Words: Set, Graph, Vertex, Edges, Domination,
M.S.C. : 05C69

Introduction:

Domination is an area in graph theory with an extensive research activity. In 1998, a book [13] on domination has been published which lists 1222 papers in this area. In general, a dominating set in a graph is a set of vertices D such that each vertex is either in D or is adjacent to a vertex in D. We will show all fundamental definitions in this paper. The historical roots of domination is said to be the following chess problem. Consider an 8 x 8 chessboard on which a queen can move any number of squares vertically, horizontally, or diagonally. Figure 1 shows the squares that a queen can attack or dominate. One is interested to find the minimum number of queens needed on the chessboard such that all squares are either occupied or can be attacked by a queen. In Figure 2, five queens are shown who dominate all the squares.

To model the queens' problem on a graph, let G represent the chessboard such that each vertex corresponds to a square, and there is an edge connecting two vertices if and only if the corresponding squares are separated by any number of squares horizontally, vertically, or diagonally. Such a set of queens in fact represents a dominating set.

For another motivation of this concept, consider a bipartite graph where one part represents people, the other part represents jobs, and the edges represent the skills of each person. Each person may take more than one job

		X					X
		X				X	
		X			X		
X		X		X			
	X	X	X				
X	X	Q	X	X	X	X	X
	X	X	X				
X		X		X			

figure 1: squares attacked by a queen.

								Q
						Q		
					Q			
				Q				
	Q							

figure 2: 5 dominating queens.

One is interested to find the minimum number of people such that are jobs are occupied. As shown in Figure 3, {X, W} form a minimum size dominating set. The concept of dominating set occurs in variety of problems. The puzzles above are only interesting examples. A number of these problems are motivated by communication network problems, for example. The communication network includes a set of nodes, where one node can communicate with another if it is directly connected to that node. In order to send a message directly from a set of nodes to all others, one needs to choose this set such that all other nodes are connected to at least one node in the set. Now, such a set is a dominating set in a graph which represents the network. For other applications of domination, the facility location problem, land surveying, and routings can be mentioned.

An essential part of the motivations in this field is based on the varieties of domination. There are more than 75 variations of domination cited in [13]. These variations are mainly formed by imposing additional conditions on D, V(G) — D, or V(G).

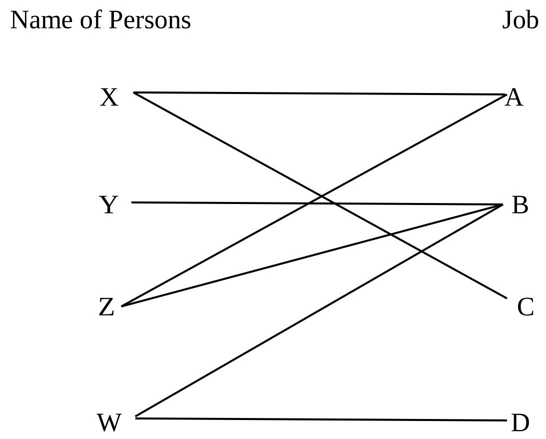

figure 3: dominated jobs.

We will review some of these variations in the next chapter, but our main focus is two type of domination, namely, factor domination and distance domination.

Basic Definitions:

We cover some basic definitions and notations here. We will define others when necessary..

A graph G = (V, E) consists of a vertex set V and edge set E. Let n = |V(G)| denote the order of G. In a graph G, the degree of a vertex v is the number of vertices adjacent to v, denoted by $d_G(v)$. The minimum and maximum degree of a graph are denoted by $\delta(G)$ and $\Delta(G)$ respectively. A vertex v is an isolated vertex if and only if $d_G(v)= 0$. A graph is connected if for every pair of vertices u and v there is a u — v path in the graph. If G is connected, then the distance between two vertices u and v is the minimum length of a u — v path in G, denoted by $d_G(u, v)$.

Let $N_G(v)$ denote the set of neighbours of a vertex $v \in V(G)$, and let $N_G[v] = N_G(v) \cup \{v\}$ be the closed neighbourhood of v in G. Let $d_G[v] = |N_G[v]| = d_G(v) + 1$.

Domination:

A dominating set D is a set of vertices such that each vertex of G is either in D or has at least one neighbour in D. The minimum cardinality of such a set is called the domination number of G, $\gamma(G)$. In Figure 4 filled vertices form a minimum size dominating set in the Petersen line graph. Therefore, $\gamma(L(P)) = 3$.

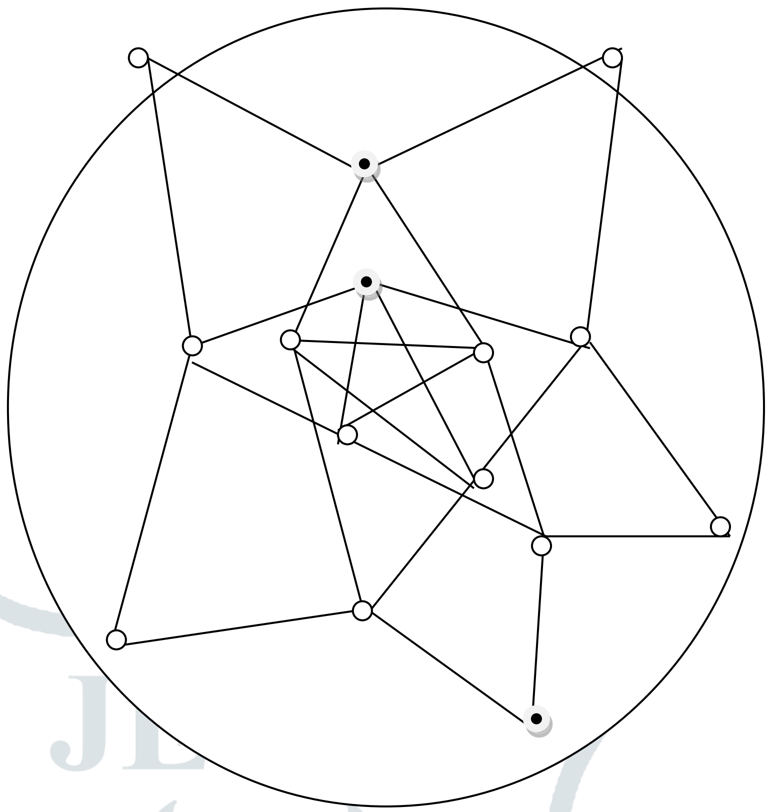

figure 4: a minimum dominating set in l(p).

The problem of determining the size of a minimum dominating set is NP-complete [8]. Actually the problem remains NP-complete even when restricted to certain classes of graphs such as bipartite graphs and chordal graphs. However, there are interesting classes of graphs such as trees, interval graphs, and cographs for which $\gamma(G)$ can be computed in polynomial time.

We will also concentrate on bounds on the domination number $\gamma(G)$ in terms of order, maximum, and minimum degree of G, all of which have been studied widely. It can be seen directly from the definition that $1 \leq \gamma(G) \leq n$. The following examples show that this bound is sharp. Let G be a graph with A(G) = n-1, then the vertex of maximum degree dominates all other vertices in G and therefore $\gamma(G) = 1$. For the lower bound, let G be an edgeless graph, then the dominating set must contain all the vertices, and $\gamma(G) = n$.

In the following trivial argument, better bounds on $\gamma(G)$ in terms of order And degrees of vertices of G can be obtained. Let v be a vertex of maximum degree in G. Since v dominates itself and all vertices in its neighbourhood, A(G) + 1 vertices are dominated by v and the trivial upper bound follows. For the lower bound, since each vertex can dominate at most A (G) other vertices and itself, the lower bound follows.

Types of Domination:

Factor and Global Domination

In order to be able to explain these concepts, we need some definitions. A factor of G is a spanning subgraph of G. A k-factoring of G is a set of k factors $f = \{G_1, G_2, ...G_k\}$ whose union is G. Figure 5 and Figure 6 show a 2-factoring of C_3P_5. A factor dominating set with respect to f is a set of vertices D which is a dominating set in each factor G_i, for $1 \leq i \leq k$. The minimum cardinality of a factor dominating set with respect to f is called the factor domination number γ (G, f). In Figure 7, filled vertices form a minimum size factor dominating set for the given 2-factoring. Hence, γ (C_3P_5, f) = 7 when f is as shown.

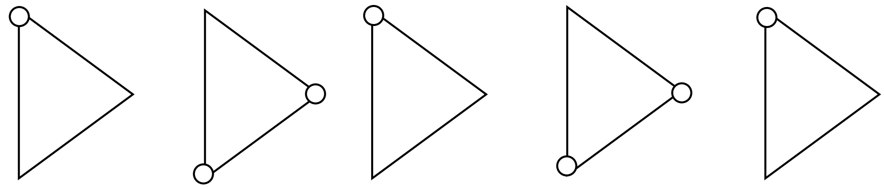

figure 5:G_1.

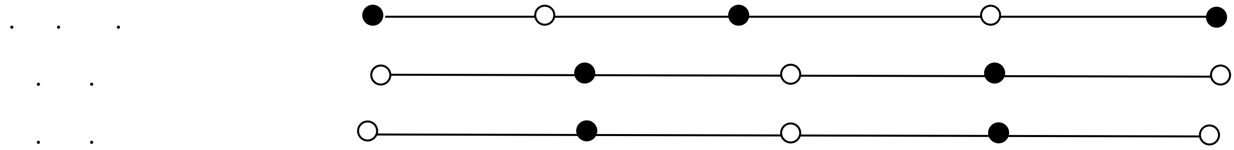

figure 6: G_2

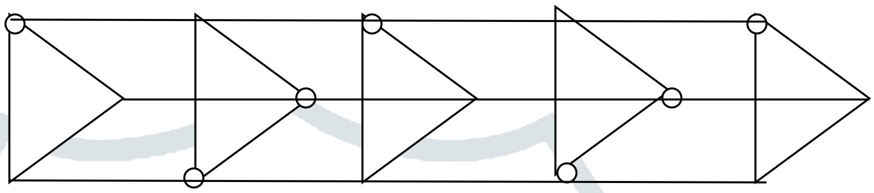

figure 7: A minimum factor dominating set for a given 2-factoring of C_3P_5.

Let $\delta_{min} = \min\{d_{Gi}(v): v \in V(G), 1\leq i\leq k\}$ be the smallest minimum degree among all factors G_1........G_k and let $\Delta_{max} = \max\{d_{Gi}(v): v \in V(G), 1\leq i\leq k\}$ be the largest maximum degree over all factors G_1........G_k

Factor domination has several applications in many network commu-nication problem such as the following: How can one send a message from a subset of nodes of a network and have it received after one hop by all other nodes using only links of some private sub network for security or redundancy reasons. To model this problem, let the communication network be represented by a graph G where vertices of G correspond to nodes of the network and edges correspond to links joining nodes which can communicate directly, and finally k edge-disjoint factors of G represent k private sub networks.

Therefore, the factor domination number represents the minimum number of nodes needed to send a message from, such that all other nodes receive the message in each sub network independently in one hop.

Now we will see with this approach to factor domination problem by restricting k, the number of factors. Brigham and Dutton [12], and also Sampathkumar independently introduced the concept of global domination which is the case of factor domination of Kn with a 2-factoring. In other words, a global dominating set is a dominating set in both G and O. The minimum cardinality of a global dom-inating set is called global domination number $\gamma_g(G)$. There are many bounds on global domination number $\gamma_g(G)$ in terms of many graphical invariants, beside those obtained from $\gamma(Kn, f)$ by setting $k = 2$.

Distance Domination

The concept of domination can be extended into distance version which is more applicable in practical problems. For example consider the communication network problem. Here, a transmitting group is a subset of those cities that are able to transmit messages to every city in the network, via communication links, by at most l hops.

An 1-dominating set is a set D_l such that every vertex in $V — D_l$ is within distance1 from at least one vertex in D_l. The minimum cardinality of a distance dominating set is called 1-domination number $\gamma_l(G)$. In Figure 8, filled vertices form a 3-dominating set of minimum size in P_5P_7. Hence, γ_3 $(P_5P_7) = 3$.

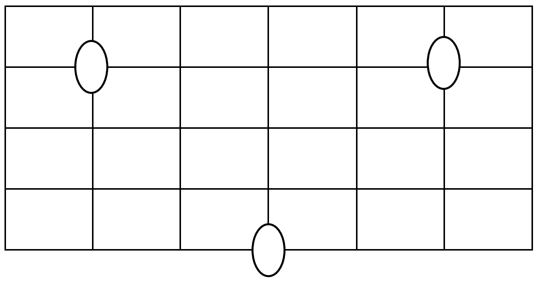

figure 8: A minimum 3-dominating set in P_5P_7.

The problem of finding l-dominating sets of relatively small sizes is important in a variety of contexts such as problems of placement of the minimum number of objects (hospitals, police,etc) within desired distance of a given population.

The concept of distance domination was first introduced by Meir and Moon. The problem of finding $\gamma_l(G)$ appears to be very difficult and only few results are known. This problem is NP-complete as shown in [5]. The restriction of the problem to special classes of graphs such as bipartite or chordal graphs of diameter 2r + 1, has been studied and the problem remains NP-complete [5]. This motivated research on achieving good bounds for distance domination

Connected Domination

Connected dominating set is a dominating set which induces a connected subgraph. Since each dominating set has at least one vertex in each component of G, only connected graphs have a connected dominating set. Therefore, here we may assume all graphs are connected. The minimum cardinality of a connected dominating set is called connected domination number $\gamma_c(G)$. In Figure 9 filled vertices form a connected dominating set of minimum size and therefore, $\gamma_c(G)$= 3.

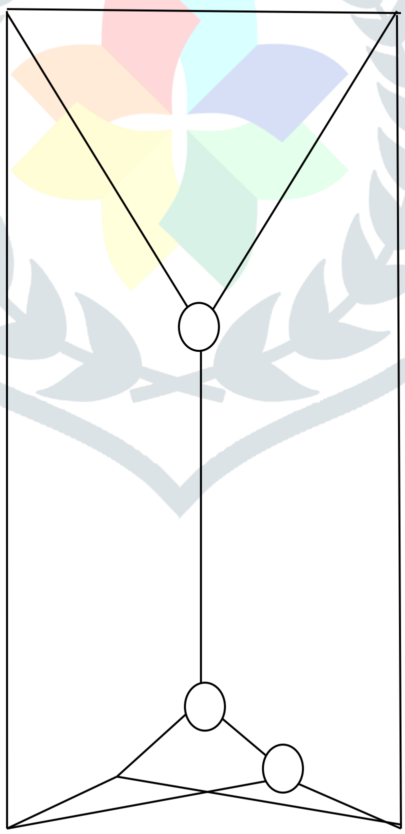

figure 9: A minimum connected dominating set.

A direct application of connected domination is again in the computer networks where a connected dominating set serves as a communication backbone in network. For example in a cellphone network.

The concept of connected domination was introduced by Sampathkumar and Walikar [4] and they showed that if H is a connected spanning subgraph of G, then $\gamma_c(G) \leq \gamma_c(H)$, since every connected dominating set of H is also a connected dominating set of G.

It has been shown that the problem of deciding whether a connected dominating set of size M exist such that $\gamma_c(G) \geq M$ is NP-complete in [8]. This problem is equivalent to the problem of finding a spanning tree that maximizes the number of leaves, since a set is a connected dominating set if and only if its complement is a set of leaves of a spanning tree. Considering this, Hedetniemi and Laskar showed that for any graph of order $n \geq 3$, $\gamma_c(G) \leq n-2$, since any tree T has at least 2 leaves.

Kleiman and West [2] studied the number of leaves in spanning trees of a connected graph and they showed that if G is a connected graph with $\delta(G) \geq k$, then it has a spanning tree with at least n – 3 [n/k+1] + 2 leaves. As discussed above, the complement of this set of leaves is a connected dominating set.

Dominating Cycles:

A dominating cycle is a cycle in which every vertex is in the neighbourhood of a vertex on the cycle. The minimum cardinality of a dominating cycle is denoted by $\gamma_{cy}(G)$. In Figure 10, filled vertices form a dominating cycle of minimum size and hence, $\gamma_{cy}(G) = 3$.

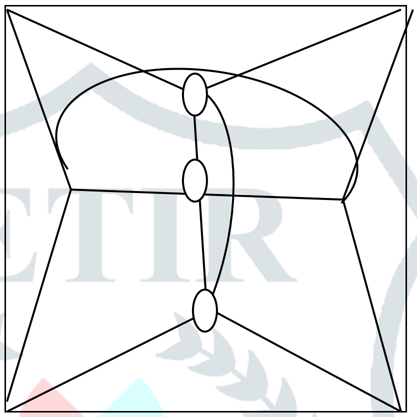

figure 10: A minimum dominating cycle.

Lesniak and Williamson [6] introduced the concept of dominating cycles. The problem is shown to be NP-complete even when restricted to planar graphs. However, there are polynomial time algorithms for few classes of graphs such as circular-arc graphs.

It is obvious that not all graphs have dominating cycles. Therefore, one must see sufficient conditions for the existence of a dominating cycle also.

Total domination

Total dominating set is a set of vertices such that each vertex $v \varepsilon V$ is in open neighbourhood of a vertex in the set. Note that in total domination vertex v does not dominate itself and so it is required that there be no isolated vertex. The minimum cardinality of a total dominating set is called the total domination number $\gamma_t(G)$. The decision problem to determine the total domination number of a graph is known to be NP-complete.

Paired domination

Paired dominating set is a dominating set whose induced subgraph has a perfect matching. From the definition it requires that there be no isolated vertices. Every paired dominating set is a total dominating set. The paired domination number is the minimum cardinality of a paired dominating set and denoted by $\gamma_{pr}(G)$. The paired dominating set problem is also shown to be NP-complete [14]. Haynes and Slater presented the following sharp bounds on $\gamma_{pr}(G)$.

k-domination

A k-dominating set is a set of vertices D such that each vertex in V(G) - D is dominated by at least k vertices in D for a fixed positive integer k. The minimum cardinality of a k-dominating set is called k-domination number $\gamma_k(G)$. In k-domination Fink and Jacobson presented the upper bound in terms of order, maximum degree and k.

k-tuple domination

k-tuple domination is a variation of k-domination which was introduced by Harary and Haynes. A k-tuple dominating set is a set of vertices D such that each vertex in V is dominated by at least k vertices in D. Therefore $k \leq \delta(G)+1$.The k-tuple domination number is the minimum cardinality of a k-tuple dominating set and denoted by $\gamma_{xk}(G)$. It is obvious that a graph C has a k-tuple dominating set if and only if $\delta \geq k — 1$.

The main application of k-tuple domination in network is for fault tolerance or mobility. Where, a node can use a service only if it is replicated on it or its neighbourhood. So, each node needs to have k copies of the service available in its closed neighbourhood.

Domatic Number and Variations

An interesting variant of domination problem is to ask how many dominating sets one can pack into a given graph G. Such type of packing questions are common for many problems and we will survey some results in this direction, mostly related to domination parameters . The main question is how to partition the vertex set of a graph into maximum number of disjoint dominating sets. The word "domatic" is created from two words "dominating" and "chromatic" since the definition of it is related to both domination and colouring concepts.
A domatic partition is a partition of V(G) such that each class of the partion is a domainating set in G. The maximum number of classes in a domatic partition is called the domatic number of G, denoted by D(G).In any graph G, {V(G)} is a domatic partion and therefore, $D(G) \geq 1$. This bound is sharp since for any graph G, with an isolated vertex, D(G) = 1.It also follows from the definitions of domination number and domatic number of a graph that $\gamma(G) \bullet D(G) \leq n$, therefore, $D(G) \leq n/\gamma(G)$

The concept of domatic partition arises in various areas. In particular, in the problem of communication networks. Domatic number of a graph represents the maximum number of disjoint transmitting groups. Another application of domatic number is related to the task of distributing resources in a computer network in the most economic way. Suppose, for example, resources are to be distributed in a computer network in such a way that expensive services are quickly accessible in the neighbourhood of each vertex. If every vertex can serve a single resource only, then the maximum number of resources that can be supported equals the domatic number in the graph representing the network.

The concept of domatic number was introduced by Cockayne and Hedetniemi in [3].

Conclusion:

In this paper we have attempted to categorize domination concepts into Some categories. For a nontrivial connected graph G = (V;E), if we use V for the set of vertices, and E for the set of edges of G, then the domination concept is shown and we will try to show types of donation we use while studying the domination theory in graph. The concept of Domination number k(G) is also discussed.

References:

[1]. D. Archdeacon, J. Ellis-Monaghan, D. Fisher, D. Froncek, P. C. B. Lam, S. Seager, B. Wei, and R. Yuster. Some remarks on domination. J. Graph Theory, 46(3):207-210, 2004.
[2]. Daniel J. Kleitman and Douglas B. West. Spanning trees with many leaves. SIAM J. Discrete Math., 4(1):99-106, 1991.
[3]. E. J. Cockayne and S. T. Hedetniemi. Towards a theory of domination in graphs. Networks, 7(3):247-5261, 1977.
[4]. E. Sampathkumar and H. B. Walikar. The connected domination number of a graph. J. Math. Phys. Sci., 13(6):607-613, 1979.
[5]. Gerard J. Chang and George L. Nemhauser. The k-domination and k-stability problems on sun-free chordal graphs. SIAM J. Algebraic Discrete Methods, 5(3):332-345, 1984
[6]. Linda M. Lesniak-Foster and James E. Williamson. On spanning and dominating circuits in graphs. Canad. Math. Bull., 20(2):215-220, 1977
[7]. Lucia D. Penso and Valmir C. Barbosa. A distributed algorithm to find k-dominating sets. Discrete Appl. Math., 141(1-3):243-253, 2004.
[8]. Michael R. Carey and David S. Johnson. Computers and intractability. W. H. Freeman and Co., San Francisco, Calif.,1979. A guide to the theory of NP-completeness, A Series of Books in the Mathematical Sciences
[9]. N. Sridharan, V. S. A. Subramanian, and M. D. Elias. Bounds on the distance two-domination number of a graph. Graphs Combin., 18(3):667-675,2002
[10]. Noga Alon, Guillaume Fertin, Arthur L. Liestman, Thomas C. Shermer, and Ladislav Stacho. Factor d-domatic colorings of graphs. Discrete Math., 262(1-3):17-25, 2003.

[11]. P. Dankelmann and R. C. Laskar. Factor domination and minimum degree. Discrete Math., 262(1-3):113-119, 2003.

[12]. Robert C. Brigham and Ronald D. Dutton. Factor domination in graphs.Discrete Math., 86(1-3):127-136,1990

[13]. Teresa W. Haynes and Peter J. Slater. Paired-domination in graphs.Networks, 32(3):199-206,1998

[14]. Teresa W. Haynes and Stephen T. Hedetniemi, editors. Domination in graphs, volume 209 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker Inc., New York, 1998. Advanced topics.

[15]. Teresa W. Haynes, Stephen T. Hedetniemi, and Peter J. Slater. Fundamentals of domination in graphs, volume 208 of Monographs and Textbooks in Pure and Applied Mathematics. Marcel Dekker Inc., New York, 1998.

[16]. Yair Caro, Douglas B. West, and Raphael Yuster. Connected domination and spanning trees with many leaves. SIAM J. Discrete Math.,13 (2):202-211 (electronic), 2000.