A Batch Arrival Queue with an Additional Multiphase Service Channel, Setup and Unreliable Server

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Abstract

In this paper, we deal with a single non reliable server $M^x/G/1$ queue with multiphase service and setup. The customers arrive in batches according to a Poisson process. Two types of services are provided to the customers, the first "essential" service and second multiphase "optional" service. After the completion of the essential service, the customer either leaves the system with probability $(1-r_1)$ or join the first optional service with probability r_1; again after completing the first phase optional service, either he leaves or joins second phase of optional service with probability r_2 and similarly in continuation at the end of $(k-1)^{th}$ phase optional service, he may opt k^{th} phase of optional service with probability r_k or may leave the system with probability $(1-r_k)$. Both essential and optional services are provided by same single server. While the server is working, he is subject to breakdown according to Poisson process. When the server breaks down, he requires repair at repair facility where a repairman renders repair of failed server according to general distribution. By introducing supplementary variable technique and generating function method, some queueing and reliability characteristics of the system are derived. We facilitate numerical results to illustrate the effect of different parameters on several performance indices.

***Key-words**: Batch arrivals, Unreliable server, Setup time, Multiphase optional service, Supplementary variable, Generating function, Queue size, Reliability.*

Introduction

During the last few decades considerable attention has been paid to studying the batch arrival queue, which has been well documented because of its interdisciplinary character in queueing systems. A single server batch arrival queue with returning customers has been proposed by Falin (2010).The evolution process of queues at signalized intersections under batch arrivals is considered by Yang and Shi (2018). A variable service speed single server queue with batch arrivals and general setup times is analyzed by Yajima and Phung-Duc (2020).

Optional phase service systems have been discussed in the literature for their application in various areas such as computer, communication, manufacturing and other many systems. These queueing systems are characterized by the feature that all arrivals demand the first essential service, whereas only some of them demand second optional service which is provided by the same server.. Functional analysis method The M/G/1 queueing model with optional second service is studied by Gupur and Kasim (2014) by functional analysis method.

Classical studies on queueing systems use perfect (reliable) servers. However in many real time systems, the server may meet unpredictable breakdowns. Therefore, queueing models with server breakdowns are realistic representation of the system. Performance analysis of bulk arrival queue is made by Singh et. Al. (2018) with balking, optional service, delayed repair and multiphase repair

The purpose of this work is to obtain explicit expressions for various queueing and reliability indices for unreliable server queue with bulk arrival. We describe the model and introduce some notations in section 2. Section 3 is devoted for the analysis part of the problem, where we obtain probability distribution of the system state. These results are obtained by the method of supplementary variable. Since the breakdowns and repair process are independent of the servicing process, then the reliability and availability are defined in the usual way in section 4. Some concluding remarks are outlined in last section 5.

2. Model Description

$M^x/G/1$ queueing system with unreliable server, setup and k-phase optional service is considered by making the following assumptions:

- The customers arrive at the system according to a compound Poisson process with random batch size denoted by random variable 'X' with distribution $a_i=Pr[X=i]$.
- There is a single unreliable server who provides two kinds of general heterogeneous services to the customers on a first come first served (FCFS) basis.
- The first essential service is needed to all arriving customers; the duration of essential services are general distributed. Its distribution function, density function and hazard rate function are $B_0(x)$, $b_0(x)$ and $\mu_0(x)$, respectively.
- As soon as the first essential service of the customer is completed, then with probability r_1 he may demand for first phase second optional service or may leave the system with probability $(1-r_1)$. After the completion of first phase optional service he may go for second phase optional service with probability r_2 or may leave the system with probability $(1-r_2)$. In general, the customer may opt any of k^{th} $(1 \le k \le m)$ phase optional service with probability r_k or may leave the system with probability $(1-r_k)$.
- The k type second optional service time follows an arbitrary distribution and its distribution function, density function and hazard rate function are $B_k(x)$, $b_k(x)$ and $\mu_k(x)$, respectively $(1 \le k \le m)$.
- We assume that the life time of a server is exponentially distributed with rate α_1 and α_2 in first essential service and second optional service, respectively.
- If the server breaks down during the service, the customer just being served before server breakdown waits for the server to complete its remaining service.
- The repair time distributions for both essential and k^{th} optional service phases are arbitrarily distributed with probability distribution functions $R_0(y)$ and $R_k(y)$, respectively. Also let $r_0(y)$, $r_k(y)$ and $\beta_0(y)$, $\beta_k(y)$ are the corresponding probability density functions and hazard rates.
- The server will be recovered after completion of the repair and starts service of the customers immediately.

Notations

λ	Mean arrival rate of the customers
X	Random variable denoting the batch size
$X(z)$	Generating function for batch size X
α_0, α_k	Mean failure rate of server in both phases, k=1,2,…,m
μ_0, θ_0, β_0	Service rate, setup rate and repair rate in first essential service
μ_k, θ_k, β_k	Service rate, setup rate and repair rate in k^{th} phase (k=1,2,…,m) second optional service
$\mu_0(x), \theta_0(y), \beta_0(y)$	Hazard rates of service, setup and repair for essential service
$\mu_k(x), \theta_k(y), \beta_k(y)$	Hazard rates of service, setup and repair for optional service
$b_0(x), s_0(y), r_0(y)$	Probability density functions for service time, setup time and repair time in essential service
$b_k(x), s_k(y), r_k(y)$	Probability density functions for service time, setup time and repair time in k^{th} phase optional service, k=1,2,…,m
$B_0(x), S_0(y), R_0(y)$	Distribution functions of service time, setup time and repair time for essential service
$B_k(x), S_k(y), R_k(y)$	Distribution functions of service time, setup time and repair time for k^{th} phase optional service, k=1,2,…,m
$P_n^{(0)}(t,x)$	Joint probability that there are n customers in the queue at time t when the server is busy with first essential service and elapsed service time lies in (x, x+dx)
$S_n^{(0)}(t,x,y)$	Joint probability that there are n customers in the queue at time t when the server is in setup state while broken down during first essential service and the elapsed

service time for the customer under service is equal to x, elapsed setup time lies in (y, y+dy)

$R_n^{(0)}(t,x,y)$ Joint probability that there are n customers in the queue at time t when the server is under repair state while broken down during first essential service and the elapsed service time for the customer under service is equal to x, elapsed repair time lies in (y, y+dy)

$P_n^{(k)}(t)$ Joint probability that there are n customers in the queue at time t when the server is busy with k[th] phase optional service, k=1,2,..,m

$S_n^{(k)}(t,y)$ Joint probability that there are n customers in the queue at time t when the server is in setup state while broken down during k[th] phase optional service and elapsed setup time lies in (y, y+dy), k=1,2,…,m

$R_n^{(k)}(t,y)$ Joint probability that there are n customers in the queue at time t when the server is in setup state while broken down during k[th] phase optional service and elapsed repair time lies in (y, y+dy), k=1,2,…,m

Hazard rates are given by:

$$\mu_k(x)dx = \frac{dB_k(x)}{1-B_k(x)};\ \theta_k(y)dy = \frac{dS_k(y)}{1-S_k(y)};$$

$$\beta_k(y)dy = \frac{dR_k(y)}{1-R_k(y)},$$

In order to provide analytic solution, the following probability generating functions are defined

$$X(z) = \sum_{i=1}^{\infty} a_i z^i,\ P^{(0)}(x,z) = \sum_{n=0}^{\infty} P_n^{(0)}(x) z^n,\ P^{(k)}(z) = \sum_{n=0}^{\infty} P_n^{(k)} z^n,$$

$$S^{(0)}(x,y,z) = \sum_{n=0}^{\infty} S_n^{(0)}(x,y) z^n,\ S^{(k)}(y,z) = \sum_{n=0}^{\infty} S_n^{(k)}(y) z^n,$$

$$R^{(0)}(x,y,z) = \sum_{n=0}^{\infty} R_n^{(0)}(x,y) z^n,\ R^{(k)}(y,z) = \sum_{n=0}^{\infty} R_n^{(k)}(y) z^n$$

3. The Analysis

We construct the partial differential equations governing the model for the system and assume the elapsed service time, elapsed setup time and the elapsed repair time as supplementary variables:

$$\left(\frac{d}{dt}+\lambda\right)Q(t) = \mu_k P_0^{(k)}(t) + (1-r_k)\int_0^{\infty} P_0^{(0)}(t,x)\mu_0(x) \qquad (1)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_0(x)+\lambda+\alpha_0\right)P_n^{(0)}(t,x) = \lambda\sum_{i=1}^{n} a_i P_{n-i}^{(0)}(x) + \int_0^{\infty} R_n^{(0)}(t,x,y)\beta_0(y)dy,\quad n\ge 1 \qquad (2)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial w}+\theta_0(y)+\lambda\right)S_n^{(0)}(t,x,y) = \lambda\sum_{i=1}^{n} a_i S_{n-i}^{(0)}(t,x,y), \qquad (3)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\beta_0(y)+\lambda\right)R_n^{(0)}(t,x,y) = \lambda\sum_{i=1}^{n} a_i R_{n-i}^{(0)}(t,x,y),\quad n\ge 1 \qquad (4)$$

$$\left(\frac{d}{dt}+\mu_k+\lambda+\alpha_k\right)P_n^{(k)}(t) = \lambda\sum_{i=1}^{n} a_i P_{n-i}^{(k)}(t) + \int_0^{\infty} R_n^{(k)}(t,y)\beta_k(y)dy + r\int_0^{\infty} P_n^{(0)}(t,x)\mu_0(x)dx,\quad n\ge 1,\ 1\le k\le m \qquad (5)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial w}+\theta_k(y)+\lambda\right)S_n^{(k)}(t,y) = \lambda\sum_{i=1}^{n} a_i S_{n-i}^{(k)}(t,y),\quad n\ge 1,\ 1\le k\le m \qquad (6)$$

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial y}+\beta_k(y)+\lambda\right)R_n^{(k)}(t,y) = \lambda\sum_{i=}^{n} a_i R_{n-i}^{(k)}(y),\quad n\ge 1,\ 1\le k\le m \qquad (7)$$

The following boundary conditions are taken into consideration:

$$P_n^{(0)}(t,0) = \mu_k P_{n+1}^{(k)}(t) + (1-r_k)\int_0^{\infty} P_{n+1}^{(0)}(t,x)\mu_0(x)dx, \qquad (8)$$

$$P_0^{(0)}(t,0)=\mu_k P_1^{(k)}(t)$$
$$+(1-r_k)\int_0^\infty P_1^{(0)}(t,x)\mu_0(x)dx+\lambda\sum_{i=1}^{n}a_i Q_{n-i}(t),$$

(9)

$$S_n^{(0)}(t,x,0)=\alpha_0 P_n^{(0)}(t,x),\quad n\ge 1,\ 1\le k\le m$$

(10)

$$S_n^{(k)}(t,0)=\alpha_k P_n^{(k)}(t),\quad n\ge 1,\ 1\le k\le m$$

(11)

$$R_n^{(0)}(x,0)=\int_0^\infty S_n^{(0)}(t,x,y)\theta_0(y)dy \qquad (12)$$

$$R_n^{(k)}(0)=\int_0^\infty S_n^{(k)}(y)\theta_k(y)dy,\quad n\ge 1,\ 1\le k\le m$$

(13)

$$P_n^{(k)}(t,0)=\int_0^\infty P_n^{(k-1)}(t,x)\mu_{k-1}(x)dx,\quad n\ge 1,\ 1\le k\le m$$

(14)

$$S_n^{(k)}(t,0)=\int_0^\infty S_n^{(k-1)}(t,y)\theta_{k-1}(y)dy,\ n\ge 1,\ 1\le k\le m$$

(15)

$$R_n^{(k)}(t,0)=\int_0^\infty R_n^{(k-1)}(t,y)\beta_{k-1}(y)dy,\quad n\ge 1,\ 1\le k\le m$$

(16)

Taking Laplace transform of eqs (1)-(7) with respect to t, we get

$$(s+\lambda)Q^*(s)-1=\mu_k P_0^{*(k)}(s)+(1-r_k)\int_0^\infty P_0^{*(0)}(s,x)\mu_0(x)dx$$

(17)

$$\frac{\partial}{\partial x}P_n^{*(0)}(s,x)+(s+\mu_0(x)+\lambda+\alpha_0)P_n^{*(0)}(s,x)=\lambda\sum_{i=1}^{n}a_i P_{n-i}^{*(0)}(s,x)$$
$$+\int_0^\infty R_n^{*(0)}(s,x,y)\beta_0(y)dy$$

(18)

$$\frac{\partial}{\partial w}S_n^{*(0)}(s,x,y)+(s+\theta_0(y)+\lambda)S_n^{*(0)}(s,x,y)=\lambda\sum_{i=1}^{n}a_i S_{n-i}^{*(0)}(s,x,y)$$

(19)

$$\frac{\partial}{\partial y}R_n^{*(0)}(s,x,y)+(s+\beta_0(y)+\lambda)R_n^{*(0)}(s,x,y)=\lambda\sum_{i=1}^{n}a_i R_{n-i}^{*(0)}(s,x,y)$$

(20)

$$\frac{d}{dt}P_n^{(*k)}(s)+(s+\mu_k+\lambda+\alpha_k)P_n^{(*k)}(s)=\lambda\sum_{i=1}^{n}a_i P_{n-i}^{*(k)}(s)$$
$$+\int_0^\infty R_n^{*(k)}(s,y)\beta_k(y)dy+r\int_0^\infty P_n^{*(0)}(s,x)\mu_0(x)dx$$

(21)

$$\frac{\partial}{\partial w}S_n^{*(k)}(s,y)+(s+\theta_k(y)+\lambda)S_n^{*(k)}(s,y)=\lambda\sum_{i=1}^{n}a_i S_{n-i}^{(*k)}(s,y)$$

$n\ge 1,\ 1\le k\le m$ (22)

$$\frac{\partial}{\partial y}R_n^{*(k)}(s,y)+(s+\beta_k(y)+\lambda)R_n^{*(k)}(s,y)=\lambda\sum_{i=}^{n}a_i R_{n-i}^{*(k)}(s),$$

(23)

Taking Laplace transforms of boundary conditions (8)-(16), we obtain

$$P_n^{*(0)}(s,0)=\mu_k P_{n+1}^{*(k)}(s)+(1-r_k)\int_0^\infty P_{n+1}^{*(0)}(s,x)\mu_0(x)dx$$

(24)

$$P_0^{*(0)}(s,0)=\mu_k P_1^{*(k)}(s)+$$
$$(1-r_k)\int_0^\infty P_1^{*(0)}(s,x)\mu_0(x)dx+\lambda\sum_{i=1}^{n}a_i Q^*{}_{n-i}(s)$$

(25)

$$S_n^{*(0)}(s,x,0)=\alpha_0 P_n^{*(0)}(s,x) \qquad (26)$$

$$S_n^{*(k)}(s,0)=\alpha_k P_n^{*(k)}(s) \qquad (27)$$

$$R_n^{*(0)}(s,x,0)=\int_0^\infty S_n^{*(0)}(s,x,y)\theta_0(y)dy \qquad (28)$$

$$R_n^{*(k)}(s,0)=\int_0^\infty S_n^{(*k)}(s,y)\theta_k(y)dy \qquad (29)$$

$$P_n^{*(k)}(s)=\int_0^\infty P_n^{*(k-1)}(s,x)\mu_{k-1}(x)dx \qquad (30)$$

$$S_n^{*(k)}(s,0)=\int_0^\infty S_n^{*(k-1)}(s,y)\theta_{k-1}(y)dy \qquad (31)$$

$$R_n^{*(k)}(t,0)=\int_0^\infty R_n^{*(k-1)}(s,y)\beta_{k-1}(y)dy \qquad (32)$$

Theorem 1: The Laplace Stieltjes transforms and moment generating functions when the server is in busy state, under setup state and repair state respectively, are given by

$$Q^*(s)=\frac{1}{(s+\lambda-\lambda X(z_s))} \qquad (33)$$

$$P^{*(0)}(s,x,z)=P^{*(0)}(s,0,z)e^{-\phi_0(s,z)x}(1-B_0(x))$$

(34)

$$P^{*(k)}(s,z)=\frac{rb*\phi_0(s,z)}{\mu_k+\psi_k(s,z)}P^{(0)}(s,0,z)$$

(35)

$$S^{*(0)}(s,x,y,z)=e^{-(s+\lambda-\lambda x(z))y}(1-S_0(y))S^{*(0)}(s,x,0,z)$$

(36)

$$S^{*(k)}(s,y,z)=e^{-(s+\lambda-\lambda x(z))y}(1-S_k(y))S^{*(k)}(s,0,z)$$

(37)

$$R^{*(0)}(s,x,y,z)=e^{-(s+\lambda-\lambda x(z))y}(1-R_0(y))R^{*(0)}(s,x,0,z)$$

(38)

$$R^{*(k)}(s,y,z)=e^{-(s+\lambda-\lambda x(z))y}(1-R_k(y))R^{*(k)}(s,0,z)$$

(39)

Theorem 2: The marginal generating functions are obtained as

$$P^{(0)}(s,z)=\left\{\frac{1-b^*\phi_0(s,z)}{\phi_0(s,z)}\right\}P^{(1)}(s,0,z) \qquad (40)$$

$$P^{(k)}(z)=\left\{\frac{r_k b^*\phi_0(s,z)}{\mu_k+\psi_k(s,z)}\right\}P^{*(0)}(s,0,z) \qquad (41)$$

$$S^{*(0)}(s,z)=\alpha_0\left\{\frac{1-b^*\phi_0(s,z)}{\phi_0(s,z)}\right\}\left\{\frac{1-s_0^*(1+\lambda-\lambda X(z))}{(1+\lambda-\lambda X(z))}\right\}P^{*(0)}(s,0,z)$$

(42)

$$S^{*(k)}(s,z)=\alpha_k\left\{\frac{rb^*\phi_0(s,z)}{\mu_k+\psi_k(s,z)}\right\}P^{*(0)}(s,0,z) \qquad (43)$$

$$R^{*(0)}(s,z)=\alpha_0\left\{\frac{1-b^*\phi_0(s,z)}{\phi_0(s,z)}\right\}\left\{\frac{1-s_0^*(s+\lambda-\lambda X(z))}{(s+\lambda-\lambda X(z))}\right\}\left\{\frac{1-r_0^*(s+\lambda-\lambda X(z))}{(s+\lambda-\lambda X(z))}\right\}P^{*(0)}(s,0,z)$$

(44)

$$R^{*(k)}(s,z)=\alpha_k\left\{\frac{r_k b^*\phi_0(s,z)}{\mu_k+\psi_k(s,z)}\right\}\left\{\frac{1-s_k^*(s+\lambda-\lambda X(z))}{(s+\lambda-\lambda X(z))}\right\}\left\{\frac{\prod_{n=1}^{k}(1-s_n^*(s+\lambda-\lambda X(z)))}{(s+\lambda-\lambda X(z))^n}\right\}P^{*(0)}(s,0,z)$$

(45)

where

$$P^{*(0)}(s,0,z)=\frac{(\psi_k(s,z)+\mu_k)\{(s+\lambda-\lambda X(z))Q^*(s)-1\}}{\{b^*\phi_0(s,z)-z\}\{\psi_k(s,z)+\mu_k\}-r_k b^*\phi_0(s,z)\psi_k(s,z)}$$

(46)

$$\phi_0(s,z)=\{s+\alpha_0+\lambda-\lambda X(z)-\alpha_0 s_0^*(s+\lambda-\lambda X(z))\}$$

$$\psi_k(s,z)=s+\lambda-\lambda X(z)+\alpha_k-\alpha_k s_k^*(s+\lambda-\lambda X(z))\prod_{n=1}^{k-1}r_n^*(s+\lambda-\lambda X(z)),1\le k\le m$$

Theorem 3: Probability that server is in idle, busy with both phases of service, setup and under repair, respectively are given by

$$P_I=1-\rho_0\left(1+\frac{\alpha_0}{\theta_0}+\frac{\alpha_0}{\beta_0}\right)-r\sum_{k=1}^{m}\rho_k\left(1+\frac{\alpha_k}{\theta_k}+\frac{\alpha_k}{\beta_k}\right)$$

(47)

$$P_B=\rho_0+r\sum_{k=1}^{m}\rho_k \qquad (48)$$

$$P_S=\frac{\rho_0\alpha_0}{\theta_0}+r\sum_{k=1}^{m}\frac{\rho_k\alpha_k}{\theta_k} \qquad (49)$$

$$P_R=\frac{\rho_0\alpha_0}{\beta_0}+r\sum_{k=1}^{m}\frac{\rho_k\alpha_k}{\beta_k} \qquad (50)$$

where $\rho_0=\frac{\lambda E[X]}{\mu_0}$ and $\rho_k=\frac{\lambda E[X]}{\mu_k}$

4. Reliability Analysis

In order to analyze reliability indices, we consider set up and breakdown states as absorbing states. Then using notations and assumptions as defined in sections 2 and 3, we get the following set of equations:

$$\left(\frac{d}{dt}+\lambda\right)Q(t)=\mu_k P_0^{(k)}(t)+(1-r)\int_0^\infty P_0^{(0)}(t,x)\mu_0(x)$$

(51)

$$\left(\frac{\partial}{\partial t}+\frac{\partial}{\partial x}+\mu_0(x)+\lambda+\alpha_0\right)P_n^{(0)}(t,x)=\lambda\sum_{i=1}^{n}a_i P_{n-i}^{(0)}(x)$$

(52)

$$\left(\frac{d}{dt}+\mu_k+\lambda+\alpha_k\right)P_n^{(k)}(t)=\lambda\sum_{i=1}^{n}a_i P_{n-i}^{(k)}(t)+r\int_0^\infty P_n^{(0)}(t,x)\mu_0(x)dx$$

(53)

Boundary conditions:

$$P_n^{(0)}(t,0)=\mu_k P_{n+1}^{(k)}(t)+(1-r)\int_0^\infty P_{n+1}^{(0)}(t,x)\mu_0(x)dx$$

(54)

$$P_0^{(0)}(t,0) = \mu_k P_1^{(k)}(t) + (1-r)\int_0^\infty P_1^{(0)}(t,x)\mu_0(x)dx + \lambda \sum_{i=1}^{n} a_i Q_{n-i}(t)$$

(55)

$$P_n^{(k)}(t,0) = \int_0^\infty P_n^{(k-1)}(t,x)\mu_{k-1}(x)dx, \quad n \ge 1, \ 1 \le k \le m$$

(56)

Theorem 5: The Laplace Stieltjes transforms and moment generating functions of the state probabilities are given by:

$$Q^*(s) = \frac{1}{(s+\lambda-\lambda X(z_w))} \qquad (57)$$

$$P^{*(0)}(s,x,z) = e^{-(s+\alpha_0+\lambda-\lambda X(z))x}(1-B_0(x))P^{*(0)}(s,0,z)$$

(58)

$$P^{(k)}(s,z) = \left\{ \frac{r\prod_{n=0}^{k}\mu_n(x)b^*(s+\alpha_0+\lambda-\lambda X(z))}{(s+\lambda-\lambda X(z)+\mu_k)} \right\} P^{*(0)}(s,0,z)$$

(59)

where $(s+\alpha_k+\lambda-\lambda X(z)) = \xi_k(s,z), \ 0 \le k \le m$

and z_w is the root of equation

$$x = b^*\xi_0(s,z)\left\{ (1-r) + \frac{rb^*\xi_0(s,z)\prod_{n=1}^{k}\mu_{n-1}(x)}{\xi_k(s,z)+\mu_k} \right\}$$

Proof: Taking Laplace Transform of equations (51)-(56), we get:

$$(s+\lambda)Q^*(s) - 1 = \mu_k P_0^{*(k)}(s) + (1-r)\int_0^\infty P_0^{*(0)}(s,x)\mu_0(x)dx$$

(60)

$$\frac{\partial}{\partial x}P_n^{*(0)}(s,x) + (s+\mu_0(x)+\lambda+\alpha_0)P_n^{*(0)}(s,x) = \lambda\sum_{i=1}^{n} a_i P_{n-i}^{*(0)}(s,x), \quad n \ge 1$$

(61)

$$\frac{d}{dt}P_n^{(*k)}(s) + (s+\mu_k+\lambda+\alpha_k)P_n^{(*k)}(s) = \lambda\sum_{i=1}^{n} a_i P_{n-i}^{*(k)}(s) + r\int_0^\infty P_n^{*(0)}(s,x)\mu_0(x)dx,$$

$$n \ge 1, 1 \le k \le m$$

(62)

And boundary conditions become

$$P_n^{*(0)}(s,0) = \mu_k P_{n+1}^{*(k)}(s) + (1-r)\int_0^\infty P_{n+1}^{*(0)}(s,x)\mu_0(x)dx$$

(63)

$$P_0^{*(0)}(s,0) = \mu_k P_1^{*(k)}(s) + (1-r)\int_0^\infty P_1^{*(0)}(s,x)\mu_0(x)dx + \lambda\sum_{i=1}^{n} a_i Q^*{}_{n-i}(s)$$

(64)

$$P_n^{*(k)}(s) = \int_0^\infty P_n^{*(k-1)}(s)\mu_{k-1}(x)dx, \quad n \ge 1$$

(65)

Multiplying (61) with suitable power of z and some over n and using defined generating functions, we have

$$P^{*(0)}(s,x,z) = e^{-(s+\alpha_0+\lambda-\lambda X(z))x}(1-B_0(x))P^{*(0)}(s,0,z)$$

(66)

From (63) and (64) we get

$$zP^{*(0)}(s,0,z) = (1-r_k)\int_0^\infty P^{*(0)}(s,x,z)\mu_0(x)dx + \mu_k P^{*(k)}(s,z) + \lambda X(z)Q^*(s) - (1-r_k)\int_0^\infty P^{*(0)}\mu_0(x)dx - \mu_k P_0^{*(k)}(s,z)$$

(67)

Eq. (60) gives

$$[\mu_k P^{*(k)}(s) + (1-r_k)\int_0^\infty P_0^{*(0)}(s,x)\mu_1(x)dx] = 1-(s+\lambda)Q^*(s)$$

(68)

Eqs (67) and (68) give

$$zP^{*(0)}(s,0,z) = (1-r_k)\int_0^\infty P^{*(0)}(s,x,z)\mu_0(x)dx + \mu_k P^{*(k)}(s,z) + (\lambda X(z)-\lambda-s)Q^*(s)$$

(69)

From eq. (62), we have

$$(s+\lambda-\lambda X(z)+\alpha_k+\mu_k)P^{*(k)}(s,z) = r_k\int_0^\infty P^{*(0)}(s,x,z)\mu_0(x)dx$$

(70)

By solving eqs (65) and (66) and using (70), we get

$$P^{(k)}(s,z)=\left\{\frac{r_k\prod_{n=0}^{k}\mu_n(x)b^*(s+\alpha_0+\lambda-\lambda X(z))}{(s+\lambda-\lambda X(z)+\mu_k)}\right\}P^{*(0)}(s,0,z)$$

(71)

Let $(s+\alpha_k+\lambda-\lambda X(z))=\xi_k(s,z), 0\le k\le m$

Now put the value of $P^{(k)}(s,z)$ in (67)

$$P^{*(0)}(s,0,z)$$
$$=\frac{(\xi_k(s,z)+\mu_k)\left\{(s+\lambda-\lambda X(z))Q^*(s)-1\right\}}{\{(1-r)b^*(\xi_0(s,z))-z\}\{\xi_k(s,z)++\mu_k\}+rb^*(\xi_0(s,z))\prod_{n=1}^{k}\mu_{n-1}(x)}$$

(72)

Let $P^*(s,z)=P^{*(0)}(s,z)+\sum_{k=1}^{m}P^{*(k)}(s,z)$ denote the probability generating function of the number of customers in the queue when setup and breakdown states are assumed to be absorbing states. Therefore,

$$P^*(s,z)=\frac{\left\{(1-b^*(\xi_0(s,z)))(\xi_k(s,z)+\mu_k)+rb^*(\xi_0(s,z))\prod_{n=1}^{k}\mu_{n-1}(x)\right\}}{\xi_0(s,z)\left[\{(1-r)b^*(\xi_0(s,z))-z\}\{\xi_k(s,z)++\mu_k\}\right]}$$

(73)

We can find $Q^*(s)$ by solving above equation for z=1 with the help of Rouche's theorem. Thus we get

$$Q^*(s)=\frac{1}{(s+\lambda-\lambda X(z_w))}$$

(74)

where z_w is the root of equation

$$x=b^*\xi_0(s,z)\left\{(1-r)+\frac{rb^*\xi_0(s,z)\prod_{n=1}^{k}\mu_{n-1}(x)}{\xi_k(s,z)+\mu_k}\right\}$$

(i) The Laplace Stieltjes transform of system availability (A(t)) at time t is given by

$$A^*(s)=Q^*(s)+P^{*(0)}(s,1)+\sum_{k=1}^{m}P^{*(k)}(s,1)$$

$$=\frac{1}{(s+\lambda-X(z_s))}$$
$$+\frac{\left(\frac{s}{(s+\lambda-X(z_s))}-1\right)\left\{rB^*\phi_0(s,1)\phi_0(s,1)+(\psi_k(s,1)+\mu_k)\right\}}{\phi_0(s,1)[\{B^*\phi_0(s,1)-1\}\{\psi_k(s,1)+\mu_k\}-rB^*\phi_0(s,1)\psi_k]}$$

(75)

(ii) The steady state availability of the server is given by

$$A=Q+P$$
$$=1-\rho_0\left(\frac{\alpha_0}{\theta_0}+\frac{\alpha_0}{\beta_0}\right)-r_k\sum_{k=1}^{m}\rho_k\left(\frac{\alpha_k}{\theta_k}+\frac{\alpha_k}{\beta_k}\right)$$

(76)

(iii) The Laplace Stieltjes transforms of the expected number of failures of the server in the first essential service and the second optional service up to time t ($M_0(t)$, $M_k(t)$) are given by:

$$M_0^*(s)=\int_0^\infty\alpha_0P_n^{*(0)}(s,x)dx=\alpha_0P^{*(0)}(s,1)$$
$$=\frac{\alpha_0\{1-B^*\phi_0(s,1)\}\{\psi_k(s,1)+\mu_k\}(sQ^*(s)-1)}{\phi_0(s,1)\{B^*\phi_0(s,1)-1\}\{\psi_k(s,1)+\mu_k\}-rB^*\phi_0(s,1)\psi_k(s,1)}$$

(77)

$$M_k^*(s)=\int_0^\infty\alpha_kP_n^{*(k)}(s,x)dx=\alpha_kP^{*(k)}(s,1)$$
$$=\frac{r_k\alpha_kB^*\phi_0(s,1)\}sQ^*(s)-1)}{\{B^*\phi_0(s,1)-1\}\{\psi_k(s,1)+\mu_k\}-r_kB^*\phi_0(s,1)\psi_k(s,1)}$$

(78)

(iv) The steady state failure frequency of the server is given by

$$M_f=\lim_{z\to1}\int_0^\infty\left(\alpha_0P^{*(0)}(s,z)+\sum_{k=1}^{m}P^{*(k)}(s,z)\right)dx$$
$$=\alpha_0\rho_0+r\sum_{k=1}^{m}\alpha_k\rho_k$$

(79)

(v) The Laplace transform of the reliability function R(t) of the server is given by

$$R^*(s) = Q^*(s) + \lim_{z \to 1} \int_0^\infty P^{*(0)}(s,x,z)dx + \sum_{k=1}^m P^{*(k)}(s,z)$$

$$= \frac{1}{(s+\lambda-\lambda X(z_w))}$$

$$+ \frac{\{(1-b^*(\xi_0(s,z)))(\xi_k(s,z)+\mu_k)\}}{\xi_0(s,z)\left[\{(1-r)b^*(\xi_0(s,z))-z\}\{\xi_k(s,z)++\mu_k\}\right]}$$

$$\times \{(s+\lambda-\lambda X(z))Q^*(s)-1\}$$

(80)

z_w is the root of equation

$$x = b^*\xi_0(s,z)\left\{(1-r) + \frac{rb^*\xi_0(s,z)\prod_{n=1}^{k}\mu_{n-1}(x)}{\xi_k(s,z)+\mu_k}\right\}$$

(vi)The mean time to the first failure (MTTFF) of the server is given by

$$MTTFF = \int_0^\infty R(t)dt = R^*(s), \qquad \lim_{s \to 0} sQ^*(s) = P_I$$

Therefore

$$MTTFF = Q^*(0) + \left(\rho_0 + \sum_{k=1}^m r_k\rho_k\right)$$

$$\frac{\left\{(1-b^*\alpha_0)(\alpha_k+\mu_k)+rb^*(\alpha_0)\prod_{n=1}^{k}\mu_{n-1}(x)\right\}}{\alpha_0\left[\{1-(1-r)b^*(\alpha_0)\{\alpha_k++\mu_k\}+rb^*(\alpha_0)\prod_{n=1}^{k}\mu_{n-1}(x)\right]}$$

(81)

5. Concluding Remarks

We have discussed an unreliable $M^x/G/1$ queueing system with second multiphase optional service with setup. All customers demand the first "essential" service, whereas only some of them demand the k-phase "optional" service. By using the supplementary variable method, we have modeled the system as a Markov chain to obtain the stationary queue indices and reliability measures of interest. For our model, we have been able to derive the state probabilities that we can use to calculate the commonly used relevant performance measures. Many existing queueing systems dealing with customer service problems are special cases of our model. Efforts have also been made to illustrate the system indices numerically to validate the analytical results.

The considered queueing model represents many practical problems in many manufacturing, production, and computer and communication systems etc. wherein the server is not continuously available for providing service for the customers, such as service interruptions due to server breakdowns. Our model dealt stochastically those with situations arising in daily life when a batch of customers appear in the system to get service and the service time consists of preliminary service phase followed by a second optional service phase. The model investigated in this paper is more realistic than those existing ones, since it takes the behavior of arriving customers as well as optional service rendered by the server.

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