Equations of Motion in General Relativity

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Abstract :
The strong field point particle limit is defined, the surface integral approach to the evaluation of the equations of motion is described and the scaling of the matter and field variables on an initial space-like hypersurface are discussed. The Newtonian velocity-momentum relation and the Newtonian equations of motion for extended bodies are re-visited and the equations of motion in the first post Newtonian (PN) approximation are obtained and their derivation carefully analyzed.

Keywords: Strong field, Point particle limit, Surface integral, Harmonic coordinates.

1. Introduction

It is well known that the post-Newtonian approximation to general relativity is important in analyzing a number of relativistic problems, such as the equations of motion of binary pulsars, solar-system tests of general relativity and gravitational radiation reaction. Any approximation scheme requires one or more small parameters characterizing the physical system under consideration. A typical parameter which most approximation schemes adopt is the magnitude of the metric deviation from a background metric. In particular if the background in Minkowskian space-time, and there is no other parameter, the scheme is sometimes called the post-Minkowskian approximation in the sense that the constructed space-time reduces to Minkowskian space-time in the limit in which the parameter tends to zero. This limit is called the weak field limit. In the case of the post-Newtonian approximation the background space-time is also Minkowskian space-time, but there is another small parameter taken to be the typical velocity of the system divided by the speed of light. We introduce a dimensionless parameter ε to express the slowness of the system. These two parameters (the deviation from the flat metric and the velocity) are required to have a certain relation in the following sense. As the gravitational field gets weaker, all velocities and forces characteristic of the material systems become smaller, in order to permit the weakening of gravity to remain an important effect in the system's dynamics. For example in the case of a binary system the typical velocity would be the orbital velocity $v/c \sim \varepsilon$ (where c is the speed of light in a vacuum) and the deviation of the space-time metric from the flat metric would be expressed via the Newtonian potential Φ (say). These variables are then related by $\Phi/c^2 \sim v^2/c^2 \sim \varepsilon^2$, which guarantees that the system is bounded by its own gravity.

In the post-Newtonian approximation, the equations of general relativity take the form of Newtonian's equations in an appropriate limit as $\varepsilon \to 0$. Such a limit is called Newtonian limit and it will be the basis of constructing the post-Newtonian approximation. However, the limit is not in any sense trivial since it may be thought of as two limits tied together as just described. It is also worth noting that the Newtonian limit cannot be uniform everywhere for all time. For example any compact binary system, no matter how weak the gravity between its components and how slow its orbital motion, will eventually spiral together due to backreaction from the emission of gravitational waves. As a result the effects of its Newtonian gravity will be swamped by those of its gravitational waves. This will mean that higher-order effects of the post-Newtonian approximation eventually dominate the lowest-order Newtonian dynamics and thus if the post-Newtonian approximation is not carefully constructed this effect can lead to many formal problems such as divergent integrals. It has been shown that such divergences can be avoided by carefully defining the Newtonian limit. Moreover the use of such a limit provides us with a strong indication that the post-Newtonian hierarchy is an asymptotic approximation to general relativity.

2. Newtonian limit and asymptotic Newtonian sequence

This formulation is based on the observation that any asymptotic approximation of any theory needs a sequence of solutions of the basic equations of the theory. Thus if we write the equations in abstract form as

$$E(g)=0 \qquad (1)$$

for an unknown function g, we would like to have a one-parameter (or possibly multiparameter) family of solutions,

$$E(g(\lambda))=0 \qquad (2)$$

where λ is some parameter. Asymptotic approximation then means that a function $f(\lambda)$ approximates $g(\lambda)$ to order λ^p if $|f(x)-g(x)|/\lambda^p$ as $\lambda \to 0$. We choose the sequence of solutions with appropriate properties in such a way that the properties reflect the character of the system under consideration.

We shall formulate the post-Newtonian approximation according to this general notion. We aim to have an approximation that applies to the systems whose motions are described almost by Newtonian theory. Thus we need a sequence of solutions of the Einstein equations parameterized by ε (the typical velocity of the system divided by the speed of light) which has Newtonian character as $\varepsilon \to 0$.

The Newtonian character is most conveniently described by the following scaling law. The Newtonian equations involve six variables, namely the density ρ, the pressure P, the gravitational potential Φ, and the velocity v^i, (i = 1, 2, 3)

$$\nabla^2\Phi - 4\pi\rho = 0 \qquad (3)$$

$$\partial_t\rho + \nabla_i\left(\rho\,\mathrm{v}^i\right) = 0 \qquad (4)$$

$$\rho\,\partial_t \mathrm{v}^i + \rho\,\mathrm{v}^j\nabla_j\,\mathrm{v}^i + \nabla^i P + \rho\nabla^i\Phi = 0 \qquad (5)$$

supplemented by an equation of state. For simplicity we have considered a perfect fluid. It can be seen that the variable $\rho(x^i,t)$, $P(x^i,t)$, $\Phi(x^i,t)$ and $\mathrm{v}^i(x^i,t)$ obeying the above equations satisfy the following scaling law :

$$\begin{aligned} \rho(x^i,t) &\to \varepsilon^2\rho(x^i,\varepsilon t), \\ P(x^i,t) &\to \varepsilon^4 P(x^i,\varepsilon t), \\ \mathrm{v}^i(x^k,t) &\to \varepsilon \mathrm{v}^i(x^k,\varepsilon t), \\ \Phi(x^i,t) &\to \varepsilon^2\Phi(x^i,\varepsilon t). \end{aligned} \qquad (6)$$

One can easily understand the meaning of this scaling since the parameter ε is the slowness introduced earlier. The magnitude of the gravitational potential will be of order ε^2 because of the balance between gravity and the centrifugal force. The scaling of the time variable expresses the time variable expresses the fact that the weaker gravity is ($\varepsilon \to 0$) the longer is the time scale. Thus we wish to have a sequence of solutions of Einstein's equations which has the above scaling as $\varepsilon \to 0$. We shall also take the point of view that the sequence of solutions is determined by the appropriate sequence of initial data. This latter viewpoint has a practical advantage because there will be no solutions of Einstein's equations which satisfy the above scaling even as $\varepsilon \to 0$. This is because Einstein's equations are nonlinear in the field variables, so it will not be possible to enforce the scaling everywhere in space-time. We shall therefore impose it only on the initial data for the solution of the sequence. Following a general discussion on the formulation of the post-Newtonian approximation, independent of any initial value formalism, we will then present the concrete treatment in harmonic coordinates. Harmonic coordinates are used because of their popularity and the fact that they have some advantages in the generalization to systems with strong interval gravity.

As initial data for the matter we take the same data as in the Newtonian case, namely the density ρ, the pressure P, and the coordinate velocity v^i. In applications we usually assume a simple equation of the state which relates the pressure to the density. The initial data for the gravitational field are $g_{\mu\nu}, \partial g_{\mu\nu}/\partial t$. Since the field equations of general relativity constitute an overdetermined system of partial differential equations there will be constraint equations among the initial data for the field. We shall write the free data for the field as (Q_{ij}, P_{ij}) whose explicit forms depend on the coordinate condition one assumes. In any coordinates we shall assume these data for the field vanish since we are interested in the evolution of an isolated system by its own gravitational interaction. It is expected that this choice corresponds to the absence of radiation far from the source. Thus we choose the following initial data dictated by the Newtonian scaling :

$$\begin{aligned}
&\rho\left(t=0, x^i, \varepsilon\right) = \varepsilon^2 a\left(x^i\right), \\
&P\left(t=0, x^i, \varepsilon\right) = \varepsilon^4 b\left(x^i\right), \\
&v^i\left(t=0, x^k, \varepsilon\right) = \varepsilon c^i\left(x^k\right), \\
&Q_{ij}\left(t=0, x^i, \varepsilon\right) = 0, \\
&P_{ij}\left(t=0, x^i, \varepsilon\right) = 0.
\end{aligned} \tag{7}$$

where the functions a, b, and c^i are C^∞ functions with the compact support contained entirely within a sphere of a finite radius.

Corresponding to the above data, we have a one-parameter family of space-time parameterized by ε. It may be helpful to visualize the family as fiber bundle, with base space the real line R (coordinate ε) and fiber the space-time R^4 (coordinates t, x^i). The fiber $\varepsilon = 0$ is Minkowskian space-time since it is defined by zero data. In the following we shall assume that the solutions are smooth functions of ε for small $\varepsilon \neq 0$. We wish to take the limit $\varepsilon \to 0$ along the sequence. The limit is however, not unique and is defined by giving a smooth nowhere vanishing vector field on the fiber bundle which is nowhere tangent to each fiber. The integral curves of the vector field give a correspondence between points is different fibers, namely events in different space-times with different values of ε. Bearing in mind the Newtonian scaling of the time variable in the limit, we introduce the Newtonian dynamical time

$$\tau = \varepsilon t \tag{8}$$

and define the integral curve as the curve on which τ and x^i remain constant (in the sequel the coordinates (τ, x^i)) will be referred to as the near zone coordinates). In fact if take the limit $\varepsilon \to 0$ along this curve, the orbital period of the binary system with ε = 0.01 is 10 times that of the system with ε = 0.1 as expected from the Newtonian scaling. This is what we define as the Newtonian limit.

From now on we assume the existence of a sequence of solutions constructed from the initial data and satisfying the above scaling with respect to ε. We shall call such a sequence a regular asymptotically Newtonian sequence. We have to make further mathematical assumptions about the sequence in order to carry out explicit calculations.

3. A calculation using harmonic coordinates A

To illustrate the formulation introduced we give an explicit calculation in harmonic coordinates. The reduced Einstein equations in harmonic coordinates take the form

$$\bar{g}^{\alpha\beta}\bar{g}^{\mu\nu},\alpha\beta = 16\pi\,\Theta^{\mu\alpha} - \bar{g}^{\mu\alpha},\beta\,\bar{g}^{\nu\beta},\alpha, \tag{9}$$

$$\partial_\mu\left[\bar{g}^{\mu\nu}\partial_\nu x^\alpha\right] = 0, \tag{10}$$

where

$$\bar{g}^{\mu\nu} = (-g)^{1/2} g^{\mu\nu}, \tag{11}$$

$$\Theta^{\alpha\beta} = (-g)\left(T^{\alpha\beta} + t_{LL}^{\alpha\beta}\right), \tag{12}$$

and $t_{LL}^{\mu\nu}$ is the Landau-Lifshtiz pseudotensor. In this section we shall choose an isentropic perfect fluid for $T^{\alpha\beta}$, which is enough for most applications, and thus

$$T^{\alpha\beta} = \left(\rho + \rho\prod + P\right)u^\alpha u^\beta + P g^{\alpha\beta}, \tag{13}$$

where ρ is the rest mass density, $\prod$ the internal energy, P the pressure, and u^μ the 4-velocity of the fluid with normalization

$$g_{\alpha\beta}u^\alpha u^\beta = -1 \tag{14}$$

The conservation of energy and momentum is expressed as

$$\nabla_\beta T^{\alpha\beta} = 0 \tag{15}$$

Defining the gravitational field variable as

$$h^{\mu\nu} = \eta^{\mu\nu} - (-g)^{1/2} g^{\mu\nu}, \tag{16}$$

where $\eta^{\mu\nu}$ is the Minkowski metric, the reduced Einstein equation (9) and the gauge condition (10) take the following form :

$$\left(\eta^{\alpha\beta} - h^{\alpha\beta}\right)h^{\mu\nu},\alpha\beta = -16\pi\,\Theta^{\mu\nu} + h^{\mu\alpha},\beta\,h^{\nu\beta} \tag{17}$$

$$h^{\mu\nu},\nu = 0 \tag{18}$$

Thus the characteristics are determined by the operator $\left(\eta^{\alpha\beta} - h^{\alpha\beta}\right)\partial_\alpha \partial_\beta$ and so the light cone deviates from that in the flat space-time. We my use this form of the reduced Einstein equations in the calculation of the waveform far from the source because this deviation plays a fundamental role there. However, in studying the gravitational field near the source it is not necessary to consider the deviation of light cone from the flat case and thus it is convenient to use the following form of the reduced Einstein equations :

$$\eta^{\mu\nu} h^{\alpha\beta}_{\mu\nu} = -16\pi\, \Lambda^{\alpha\beta}, \qquad (19)$$

where

$$\Lambda^{\alpha\beta} = \Theta^{\alpha\beta} + \chi^{\alpha\beta\mu\nu}_{\mu\nu} \qquad (20)$$

$$\chi^{\alpha\beta\mu\nu} = (16\pi)^{-1}\left(h^{\alpha\nu} h^{\beta\mu} - h^{\alpha\beta} h^{\mu\nu}\right) \qquad (21)$$

Equation (18) and (19) together imply the conservation law

$$\Lambda^{\alpha\beta}_{\beta} = 0 \qquad (22)$$

We shall take as our variables the set $\{\rho, P, \mathrm{v}^i, h^{\alpha\beta}\}$ with the definition

$$\mathrm{v}^i = u^i / u^0 \qquad (23)$$

The time component of the 4-velocity u^0 is determined from (14). To make a well-defined system of equations we must add the conservation law for the number density *n*, which is some function of the density ρ and pressure *P*:

$$\nabla_\alpha \left(n u^\alpha\right) = 0 \qquad (24)$$

Equations (22) and (24) imply that the flow is adiabatic. The role of the equation of state is played by the arbitrary function $n(\rho, P)$.

Initial data for the above set of equations consists of $h^{\alpha\beta}, h^{\alpha\beta},0, \rho, P$ and v^i, but not all these data are independent because of the existence of the constraint equations. Equation (18) and (19) imply the four constraint equations among the initial data for the field,

$$\Delta h^{\alpha 0} + 16\pi\, \Lambda^{\alpha 0} - \delta^{ij} h^{\alpha,0}_{i,j} = 0 \qquad (25)$$

where Δ is the Laplacian in the flat space. We shall choose h^{ij} and $h^{ij},0$ as free data solve (25) for $h^{\alpha 0}(\alpha = 0,...3)$ and (18) for $h^{\alpha 0},0$. Of course these constraints cannot be solved explicitly since $\Lambda^{\alpha 0}$ contains $h^{\alpha 0}$, but they can be solved iteratively as explained below. As discussed above, we shall assume that the free data h^{ij} and $h^{ij},0$ for the field vanish.

In carrying out the calculation we find that it is convenient to use an expression with explicit dependence on ε. The harmonic condition allows us to have such an expression in terms of the retarded integral

$$h^{\mu\nu}\left(\varepsilon, \tau, x^i\right) = 4\int_{C\left(\varepsilon,\tau,x^i\right)} d^3 y\, \Lambda^{\mu\nu}\left(\tau - \varepsilon r, y^i, \varepsilon\right)/r + h^{\mu\nu}_{H}\left(\varepsilon, \tau, x^i\right), \qquad (26)$$

where $r=\left|y^i-x^i\right|$ and $C\left(\varepsilon,\tau,x^i\right)$as the past flat light cone of the event $\left(\tau,x^i\right)$ in the space-time given by ε, truncated where it interests with initial hypersurface $\tau=0, h_H^{\mu\nu}$ is the unique solution of the homogeneous wave equation in flat space-time,

$$\eta^{\alpha\beta}h_{\alpha\beta,H}^{\mu\nu}=0 \tag{27}$$

$h_H^{\mu\nu}$ evolve from a given initial data on the $\tau=0$ initial hypersurface which are subject to the constraint equations (25). The explicit form of $h_H^{\mu\nu}$ is available via the Poisson formula

$$h_H^{\mu\nu}\left(\tau,x^i\right)=\frac{\tau}{4\pi}\oint_{\partial C\left(\tau,x^i\right)}h_{,\tau}^{\mu\nu}\left(0,y^i\right)d\,\Omega_y+\frac{1}{4\pi}\frac{\partial}{\partial\tau}\left[\tau\oint_{\partial C\left(\tau,x^i\right)}h_{,\tau}^{\mu\nu}\left(0,y^i\right)d\,\Omega_y\right] \tag{28}$$

The zero in the argument of each integrand indicates that the integrands are to be evaluated at $\tau=0$. We shall henceforth ignore the homogeneous solutions because they play no role. Because of the ε dependence of the integration region the domain of integration is finite as long as $\varepsilon\neq 0$ and the diameter increases like ε^{-1} as $\varepsilon\to 0$.

Given the formal expression (26) in terms of initial data (7), we can take the Lie derivatives and evaluate these derivatives at ε = 0. The Lie derivative is a partial derivative with respect to ε in the coordinate system for the fiber bundle given by $\left(\varepsilon,\tau,x^i\right)$. Accordingly one should convert all the time indices to τ indices. For example $T^{\tau\tau}=\varepsilon^2T^{tt}$, which is order ε^4 since $T^{tt}\sim\rho$ is order ε^2. Similarly the other components of the stress-energy tensor $T^{ti}=\varepsilon T^{ti}$ and T^{ij} are of order ε^4. In fact we find

$$4^{h^{\tau\tau}}\left(\tau,x^i\right)=4\int_{R^3}\frac{2\rho\left(\tau,y^i\right)}{r}d^3y, \tag{29}$$

$$4h^{\tau i}\left(\tau,x^i\right)=4\int_{R^3}\frac{2\rho\left(\tau,y^k\right)1\mathrm{v}^i\left(\tau,x^k\right)}{r}d^3y, \tag{30}$$

$$4^{h^{\tau i}}\left(\tau,x^k\right)=4\int_{R^3}\frac{2\rho\left(\tau,y^k\right)1\mathrm{v}^i\left(\tau,y^k\right)1\mathrm{v}^j\left(\tau,y^k\right)+4^{t_{LL}^{ij}}\left(\tau,y^k\right)}{r}d^3y, \tag{31}$$

where we have adopted the notation

$$n\,f\left(\tau,x^i\right)=\frac{1}{n!}\lim_{\varepsilon\to 0}\frac{\partial^n}{\partial\varepsilon^n}f\left(\varepsilon,\tau,x^i\right), \tag{32}$$

and

$$4^{t_{LL}^{ij}}=\frac{1}{64\pi}\left(4^{h^{\tau\tau,i}}4^{h^{\tau\tau,j}}-\frac{1}{2}\delta^{ij}4^{h^{\tau\tau,k}}4^{h^{\tau\tau}},k\right) \tag{33}$$

In the above calculation we have taken the point of view that $h^{\mu\nu}$ is a tensor field, defined by giving its components in the assumed harmonic coordinates as the difference between the tensor density $\sqrt{-g}g^{\mu\nu}$ and $\eta^{\mu\nu}$. The conservation law (22) also has its first nonvanishing derivatives at this order, which are

$$2\rho_{,\tau}+\left(2\rho_1\mathrm{v}^i\right),i=0, \tag{34}$$

$$\left(2\rho 1\mathrm{v}^i\right)_{,\tau}+\left(2\rho 1\mathrm{v}^i 1\mathrm{v}^j\right)_{,j}+4P^i-\frac{1}{4}2\rho 4h^{\tau\tau,i}=0 \qquad (35)$$

Equation (29), (34), and (35) correspond to the equations of the Newtonian theory of gravity. Thus the lowest non-vanishing derivative with respect to ε is indeed Newtonian theory, and the 1PN and 2PN equations emerge from the sixth and eighth derivatives, respectively, in the conservation law (22).

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