Poisson Generalized Inverted Exponential distribution: Model, Properties and Applications

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Abstract

In this research paper, we propose and study Poisson Generalized Inverted exponential distribution; a new distribution which is defined by using the genesis of the Reliability Estimation in Generalized Inverted Exponential distribution with progressively type II censored sample and The Kumaraswamy Poisson-G family of distribution: its properties and application. Some of the mathematical properties of the new family including mode, skewness, kurtosis, quantile function, moments, inverse probability moments and order statistics are discussed. We also discuss maximum likelihood estimation procedure for the parameter estimation and other potential application of the distribution. The flexibility and potentiality of the proposed distribution is discussed by using simulated data set.

Key words: Poisson Generalized Inverted Exponential Distribution, Order statistics, Maximum likelihood Estimation,Inverse probability weighted Moments, quantile function

1. Introduction

Over the last decade there has been a growing interest in developing more flexible probability distributions. The extended family appears to have wider adaptability in applied science, engineering, actuarial science, economics, telecommunications, life testing and many other areas due to theirflexibility to capture diverse phenomena. Modern computing technologies have contributed heavily by providing the platform to perform necessary computations even if the analytical solutions of many of these generalizations are very complicated(Aryal,et al.2019). In literature, some familiar distribution has been derived which are used in real data analysis in different areas are, Generalized Exponential-Poisson by Barreto-Souza, &Cribari-Neto (2009), Exponentiated exponential Poisson G family by Ristić&Nadarajah (2014),KumaraswamyPoisson-G Family by Ramos et al. (2015),Exponentiated generalized-G Poisson family by Aryal et al. (2017),Poission exponential –G family by Rayad et al. (2020),and others.

Over the years new families of the distributions have been proposed to generalizevarious distributions by compounding well-known distributions to provide greater flexibility in modeling data from practical viewpoint. Joshi, R. K&Kumar, V. (2020) generated a new continuous distribution having three parameters based on the half logistic-Generating family called half logistic NHE.Ünal et al. (2018) introduced Alpha Power Inverted Exponential distribution based on the inverted exponential distribution. Joshi and Dhungana (2020) introduced a new class of life-time distribution by compounding Rayleigh distribution and Exponentiated G. Poisson distribution by power transformation technique .Dhungana (2020) developed a new Poisson Inverted Exponential distribution from the Poisson family of distribution. Almarashi et al (2019) introduced Type I half logistic exponential distribution: a new extension of exponential distribution. Muhammad, &Yahaya, (2017) proposed a new two-parameter distribution called the half -Logistic Poisson distribution followed by generalized half-Poisson distribution, consisting of three parameters, with a failure rate function that can be increasing, decreasing or upside-down bathtub-shaped depending on its parameters by Muhammad, M. (2017). Gomez et al. (2014) have presented a new extension of the exponential distribution .

To develop a probability model different method exists, among them the most common method is to combine a valid probability distribution with a family of the existing distribution. The new distribution with additional parameter provides more flexibilityin modeling data. In this paper we proposed a new probability distribution by compounding the Reliability estimation in generalized inverted exponential distribution with progressively type II censored sample and Kumaraswamy Poisson-G family distribution, the so called Poisson Generalized Inverted Exponential(PGIE) distribution because it is capable of modeling monotonically increasing , decreasing and upside down hazard rates and can be viewed as suitable model for fitting skewed data which may not be properly fitted by other common distribution and can also be used for variety of problems in different areas such as industrial reliability and survival analysis.

The remainder of this paper is organized as follows. In section 2, we introduce PGIE distribution from a practical view point. The CDF, PDF, survival function and hazards rate are explicitly presented in this section. In section 3, we discuss mode, Quantile function, skewness, kurtosis, moments, inverse probability moments, order statistics and other useful expansion for the new proposed distribution.In section 4, we have discussed Maximum likelihood estimation and asymptotic distribution. In section 5,a simulated data set is analyzed for illustrative purposes and results are discussed. Finally some concluding remarks are given in section 6.

2. Model analysis of PGIE distribution

The exponential distribution has been extended in numerous ways to get new probability models for life testing problems. Let, Y follows an exponential distribution, the distribution of $X = \frac{1}{Y}$ would be an inverted exponential distribution.Krishna & Kumar (2013)Generalized Inverted Exponential (GIE) distribution having CDF is;

$$G(x) = 1 - \left(1 - e^{-\frac{\beta}{x}}\right)^{\alpha}, x > 0, \beta > 0, \alpha > 0 \qquad (2.1)$$

In literature, not only probability distribution, but also family of distribution has been derived. When a family of distributions has being derived, Chakraborty et al. (2020) defined Poisson-G Family which CDF is;

$$F(x) = \frac{1 - e^{-\lambda G(x)}}{1 - e^{-\lambda}}, x > 0, \lambda > 0 \qquad (2.2)$$

The CDF of GIE having parameter α and β in equation (2.1) is applied in equation (2.2), then CDF of new distribution **Poisson Generalized Inverted Exponential (PGIE)**with3 parameters becomes;

$$F(x) = \frac{1}{\left(1 - e^{-\lambda}\right)}\left[1 - e^{-\lambda\left\{1 - \left(1 - e^{-\frac{\beta}{x}}\right)^{\alpha}\right\}}\right]; x > 0, \alpha > 0, \beta > 0, \lambda > 0 \qquad (2.3)$$

The PDF of purposed model becomes,

$$f(x;\alpha,\beta,\lambda) = \frac{\alpha\beta\lambda}{x^2\left(1 - e^{-\lambda}\right)} e^{-\frac{\beta}{x}}\left(1 - e^{-\frac{\beta}{x}}\right)^{\alpha-1} e^{-\lambda\left\{1 - \left(1 - e^{-\frac{\beta}{x}}\right)^{\alpha}\right\}}; x > 0, \alpha > 0, \beta > 0, \lambda > 0 \quad (2.4)$$

The left hand graph in the figure illustrates the probability distribution function of PGIE with its parameter λ fixed at 2 and its parameter β> 0 taking the values 0.1, 0.3, 0.5, 0.7, 0.9 and for α> 1 assumes decreasing and unimodal shape.

Survival function

Survival function referred to as failure- time analysis, refers to the set of statistical methods used to analyze time- to-event data.

$$R(x) = 1 - F(x) = 1 - \frac{1}{\left(1 - e^{-\lambda}\right)}\left[1 - e^{-\lambda\left\{1 - \left(1 - e^{-\frac{\beta}{x}}\right)^{\alpha}\right\}}\right]; x > 0, \alpha > 0, \beta > 0, \lambda > 0 \qquad (2.5)$$

The right hand graph in the figure illustrates the survival function of PGIE distribution.The distribution exhibits decreasing survivalfunction for constant value of the parameter λ =2 and different values of α> 0 andβ> 0.The unimodal survival shape suggests that PGIE distribution may be useful for modeling survival data that exhibits decreasing death rates.

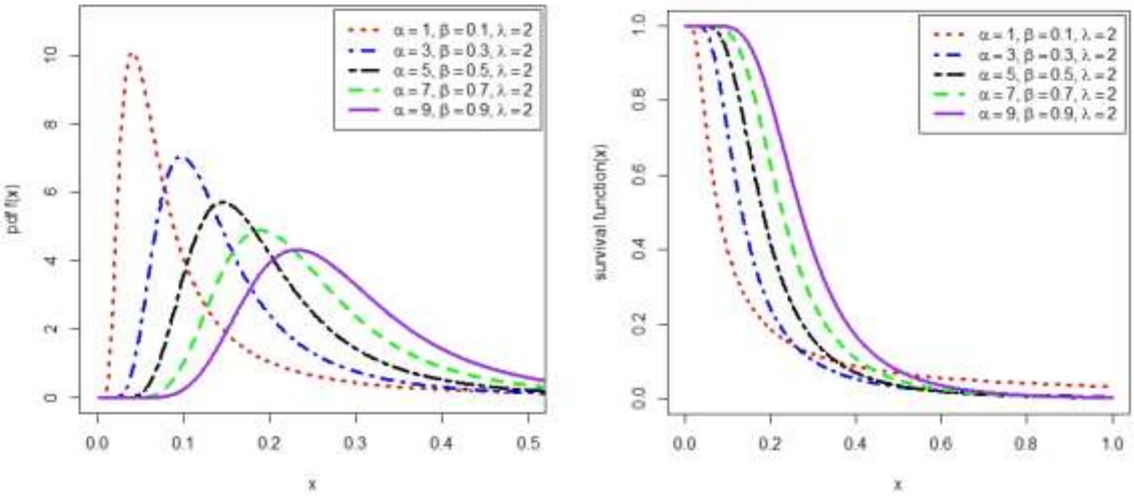

Figure 1:PDF (left panel) survival function (right panel) of proposed distribution of distribution with different parameters value

The hazard rate function is a significant quantitative exploration of life phenomenon. The hazard rate function measures the conditional instantaneous rate of failure at time x, given survival to time x. The hazard rate of PGIE distribution exhibits flexible behavior and unimodal failure rate.

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{\alpha\beta\lambda e^{-\frac{\beta}{x}}\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha-1}}{x^{2}\left[1-e^{-\lambda\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha}}\right]};x>0,\alpha>0,\beta>0,\lambda>0 \qquad (2.6)$$

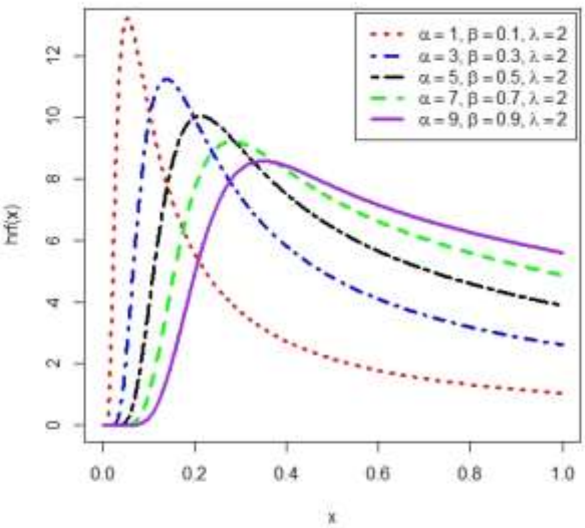

Figure 2: HRF of proposed distribution of distribution with different parameters value

The reverse hazard rate of is defined as the ratio between the life probability density to its distribution function. This concept plays a role in analyzing censored data and it is applicable in such areas as actuarial sciences, forensic studies and similar other fields. Hence, reversed –hazard rate is;

$$r(x)=\frac{f(x)}{F(x)}=\frac{\alpha\beta\lambda e^{-\frac{\beta}{x}}\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha-1}}{x^2\left[e^{\lambda\left\{1-\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha}\right\}}-1\right]};x>0,\alpha>0,\beta>0,\lambda>0 \qquad (2.7)$$

3. Statistical Properties

In this sectionthe major properties of proposed distribution are derived.

3.1 Useful expansions

For n, a positive real non integer and $|z|<1$, we have the generalized binomial series,

$$(1-x)^n=\sum_{i=0}^{\infty}(-1)^i\binom{n}{i}x^i \qquad (3.1)$$

The series corresponding to exponential function is

$$e^{-ax}=\sum_{j=0}^{\infty}\frac{(-1)^j(a\,x)^j}{j!} \qquad (3.2)$$

3.1Probability density function

Lemma. The probability density function of PGIE *is* $f(x)=\sum_{k=0}^{\infty}\frac{1}{x^2}\xi_k e^{-\frac{\beta(k+1)}{x}}$ (3.3)

Proof. The probability density function equation (2.4) of proposed distribution is

$$f(x)=\frac{\alpha\beta\lambda}{x^2\left(1-e^{-\lambda}\right)}e^{-\frac{\beta}{x}}\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha-1}e^{-\lambda\left\{1-\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha}\right\}}$$

$$=\frac{\alpha\beta\lambda}{x^2\left(1-e^{-\lambda}\right)}e^{-\frac{\beta}{x}}\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha-1}\sum_{j=o}^{\infty}\frac{(-\lambda)^j}{j!}\left\{1-\left(1-^{-\frac{\beta}{x}}\right)^{\alpha}\right\}^j \text{ apply equation (3.2)}$$

$$=\frac{\alpha\beta\lambda}{x^2\left(1-e^{-\lambda}\right)}e^{-\frac{\beta}{x}}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j}\lambda^j}{j!}\binom{j}{i}\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha(i+1)-1} \text{ apply equation (3.1)}$$

$$=\frac{\alpha\beta\lambda}{x^2\left(1-e^{-\lambda}\right)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-1)^{i+j+k}\lambda^j}{j!}\binom{j}{i}\binom{\alpha(i+1)-1}{k}e^{-\frac{\beta(k+1)}{x}}$$

$$=\sum_{k=0}^{\infty}\frac{\xi_k}{x^2}e^{-\frac{\beta(k+1)}{x}} \text{ where } \xi_k=\frac{\alpha\beta\lambda}{\left(1-e^{-\lambda}\right)}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{i+j+k}\lambda^j}{j!}\binom{j}{i}\binom{\alpha(i+1)-1}{k}$$

3.2Mode

The maximum repetitive value of the given PDF is mode. To calculate the mode the necessary and sufficient condition is; $\frac{df(x)}{dx}=0$ and $\frac{d^2f(x)}{dx^2}<0$. After applying necessary condition, we get,

$$-\frac{1}{x^4\left(1-e^{-\frac{\beta}{x}}\right)^2}\left[(2x-\beta)e^{\frac{\beta}{x}}+\alpha\beta\lambda\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha}-2x+\alpha\beta\right]\left(1-e^{-\frac{\beta}{x}}\right)^{(\alpha-1)}e^{-\lambda\left\{1-\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha}\right\}}=0 \qquad (3.4)$$

Equation (3.4) is a nonlinear equation, and its solution cannot befound analytically. It can be found numerically by using Newton-Raphson method.

3.3Quantile function

Quantile functions are used for theoretical aspects of probability theory. The quantile function is defined of any distribution is $Q(u)=F^{-1}(u)$. Therefore, the corresponding quantile function for the purposed model is;

$$Q(u)=\frac{-\beta}{\ell n\left[1-\left\{1+\frac{1}{\lambda}\ell n\left\{1-u\left(1-e^{-\lambda}\right)\right\}\right\}^{1/\alpha}\right]} \tag{3.5}$$

3.4. Skewness and Kurtosis

For statistical analysis, skewness and kurtosis are used to describe the characteristics of the distribution**.**Bowley's skewness [as cited inAl-saiary et al. (2020)] takes the form

$$S_k=\frac{Q(3/4)-2Q(1/2)+Q(1/4)}{Q(3/4)-Q(1/4)}$$

Moors' kurtosis [Al-saiary et al. (2020)] is based on Octiles and could be written as

$$K_u=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}$$

Where Q (.) is the quantile function defined in equation (3.5).

3.5 Moments

Since, $X \sim PGIE(\alpha,\beta,\lambda)$,The$r^{th}$ raw moment is defined as (used value of *f(x)* is used from equation (3.3))

$$\mu_r'=E(X^r)=\int_0^\infty x^r f(x)dx=\sum_{k=0}^\infty \xi_k\int_0^\infty x^{r-2}e^{-\frac{\beta(k+1)}{x}}dx \tag{3.6}$$

After integration of (3.6), the r^{th} raw moment of proposed model is;

$$\mu_r'=\sum_{k=0}^\infty \xi_k\frac{\Gamma(1-r)}{[\beta(k+1)]^{1-r}};\ r<1 \tag{3.7}$$

This is an indication that the rth raw moment for the proposed distribution does not exist since the expression in Equation (3.7) only exist for r< 1. Therefore the general expression for the rth inverse raw moment can be written as

$$\mu_r'=E\left(\frac{1}{X^r}\right)=\int_0^\infty \frac{1}{x^r}f(x)dx=\sum_{k=0}^\infty \xi_k\int_0^\infty \frac{1}{x^{r+2}}e^{-\frac{\beta(k+1)}{x}}dx \tag{3.8}$$

Let $\frac{1}{x}=y$ thenrth inverse raw moment can be written as $\mu_r'=\sum_{k=0}^\infty \xi_k\int_0^\infty y^r e^{-\beta(k+1)y}dy$ (3.9)

After integration of (3.9), the r^{th}inverseraw moment of proposed model is;

$$\mu_r' = \sum_{k=0}^{\infty} \xi_k \frac{\Gamma(r+1)}{[\beta(k+1)]^{r+1}}; \; r \geq 1$$

For mean, $r=1$, then $\mu_1' = \sum_{k=0}^{\infty} \xi_k \frac{1}{[\beta(k+1)]^2}$ and second r[th]inverse raw moment is

$\mu_2' = \sum_{k=0}^{\infty} \xi_k \frac{2}{[\beta(k+1)]^3}$. After that we can get inverse variance as $Var(x) = \mu_2' - (\mu_1')^2$

The lower incomplete inverse moments, say, $\varphi_s(t)$, is given by; $\varphi_s(t) = \int_0^t \frac{1}{x^s} f(x)dx$ (3.10)

Let, $\frac{1}{x} = y$ and used the relation (3.3) in equation (3.10), the equation (3.10) becomes

$$\varphi_s(t) = \sum_{k=0}^{\infty} \xi_k \int_{1/t}^{\infty} y^s e^{\beta(k+1)y} dy \qquad (3.11)$$

Now, applying upper incomplete gamma function, $\Gamma(s,t) = \int_t^{\infty} x^{s-1} e^{-x} dx$ in equation (3.11), we get the value of lower inverseincomplete moment $\varphi_s(t)$ as

$$\varphi_s(t) = \sum_{k=0}^{\infty} \xi_k \frac{\Gamma\left[(s+1), \left\{ \frac{1}{t} (1+k)\beta \right\}\right]}{[(1+k)\beta]^{(s+1)}}; s \geq 1$$

Similarly, the inverse conditional moment is defined as $\tau_s(t) = \int_t^{\infty} \frac{1}{x^s} f(x)dx$ (3.12)

Using the relation (3.3) in equation (3.12), and used $\frac{1}{x} = y$ and applying lower incomplete gamma function, $\gamma(s,t) = \int_0^t x^{s-1} e^{-x} dx$ in equation (3.12), and integrating equation (3.12), we get the value of inverse conditional moment as

$$\tau_s(t) = \sum_{k=0}^{\infty} \xi_k \frac{\gamma\left[(s+1), \left\{ \frac{1}{t} (1+k)\beta \right\}\right]}{[(1+k)\beta]^{(s+1)}}; s \geq 1$$

Likewise,Let, $y = \frac{1}{x}$,then inverse Moment Generating Function (MGF) is

$$M_Y(t) = E[e^{tY}] = \sum_{r=0}^{\infty} \frac{t^r}{r!} E\left(\frac{1}{X^r}\right) \qquad (3.13)$$

Using result of the equation (3.9) in equation (3.13), the MGF is

$$M_Y(t) = \sum_{i=0}^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} \xi_k \frac{\Gamma(r+1)}{[\beta(k+1)]^{r+1}}; \; r \geq 1$$

3.6 The Inverse Probability Weighted Moments (PWM)

The inverse probability weighted moments have been obtained from the following relation

$$\tau_{r,s} = E\left[\frac{1}{X^r}\{F(x)\}^s\right] = \int_0^\infty \frac{1}{x^r} f(x)\{F(x)\}^s dx \qquad (3.14)$$

Now, first we have determine the value of $\left[F(x)\right]^s = \frac{1}{\left(1-e^{-\lambda}\right)^s}\left[1-e^{-\lambda\left\{1-\left(1-e^{-\frac{\beta}{x}}\right)^\alpha\right\}}\right]^s ; s>1$

We apply the equation (3.1) in above relation, we get

$$= \frac{1}{\left(1-e^{-\lambda}\right)^s}\sum_{i=0}^{\infty}(-1)^i\binom{s}{i} e^{-\lambda i\left\{1-\left(1-e^{-\frac{\beta}{x}}\right)^\alpha\right\}}$$

Again apply the equation (3.2) in this relation we get

$$= \frac{1}{\left(1-e^{-\lambda}\right)^s}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{s}{i}\frac{(\lambda i)^j}{j!}\left\{1-\left(1-e^{-\frac{\beta}{x}}\right)^\alpha\right\}^j$$

$$= \frac{1}{\left(1-e^{-\lambda}\right)^s}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=o}^{\infty}(-1)^{i+j+k}\binom{s}{i}\binom{j}{k}\frac{(\lambda i)^j}{j!}\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha k}$$

$$= \frac{1}{\left(1-e^{-\lambda}\right)^s}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=o}^{\infty}\sum_{l=0}^{\infty}(-1)^{i+j+k+l}\binom{s}{i}\binom{j}{k}\binom{\alpha k}{l}\frac{(\lambda i)^j}{j!}e^{-\frac{\beta l}{x}}$$

$$= \sum_{l=0}^{\infty}\phi_l\, e^{-\frac{k\,l}{x}} \qquad (3.15)$$

Where, $\phi_l = \frac{1}{\left(1-e^{-\lambda}\right)^s}\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=o}^{\infty}(-1)^{i+j+k+l}\binom{s}{i}\binom{j}{k}\binom{\alpha k}{l}\frac{(\lambda i)^j}{j!}$

Now, using the relation (3.3) and(3.15) in equation (3.14), it becomes,

$$\tau_{r,s} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\xi_k\phi_l\int_0^\infty \frac{1}{x^{r+2}}e^{-\frac{\beta(k+l+1)}{x}}dx \qquad (3.16)$$

Let , $y=\frac{1}{x}$, then equation (3.16) becomes, $\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\xi_k\phi_l\int_0^\infty y^r e^{-\beta(k+l+1)y}dy$ (3.17)

After integrating the equation (3.17), the IPWM of proposed model is

$$\tau_{r,s} = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\xi_k\phi_l\frac{\Gamma(r+1)}{\left[\beta(k+l+1)\right]^{(r+1)}}; r\geq 1, s>1$$

3.7 Order statistics

Let $x_{(1)} < x_{(2)} < x_{(3)} < .... < x_{(n)}$ the ordered statistics of random sample of size n from the following PGIEdistribution with parameters $(\alpha, \beta \text{ and } \lambda)$ with CDF F(x) and PDF f(x). According to H. A. David (as cited in Al-Saiary et al., (2019)), the PDF of $X_{(r)}$ can be written as;

$$f_r(x_{(r)}) = \frac{n!}{(r-1)!(n-r)!} f(x)\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} \tag{3.18}$$

The expansion of $F(x) = \frac{1}{1-e^{-\lambda}} - \sum_{k=0}^{\infty} \upsilon_k e^{-\frac{\beta k}{x}}$ where $\upsilon_k = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j+k}\binom{j}{i}\binom{\alpha i}{k}\frac{\lambda^j}{j!}$

And $M = \frac{n!}{(r-1)!(n-r)!}$ then equation (3.18) becomes,

$$f_r(x_{(r)}) = M\sum_{k=0}^{\infty}\frac{\xi_k}{x_{(r)}^2} e^{-\frac{\beta(k+1)}{x_{(r)}}}\left[\frac{1}{1-e^{-\lambda}} - \sum_{k=0}^{\infty}\upsilon_k e^{-\frac{\beta k}{x_{(r)}}}\right]^{r-1}\left[1-\frac{1}{1-e^{-\lambda}} + \sum_{k=0}^{\infty}\upsilon_k e^{-\frac{\beta k}{x_{(r)}}}\right]^{n-r} \quad ; x_{(r)} > 0 \tag{3.19}$$

The PDF of largest order statistics $X_{(n)}$ is given by

$$f_n(x_{(n)}) = n\sum_{k=0}^{\infty}\frac{\xi_k}{x_{(n)}^2} e^{-\frac{\beta(k+1)}{x_{(n)}}}\left[\frac{1}{1-e^{-\lambda}} - \sum_{k=0}^{\infty}\upsilon_k e^{-\frac{\beta k}{x_{(n)}}}\right]^{n-1} \quad ; x_{(n)} > 0$$

The PDF of smallest order statistics $X_{(1)}$ is given by

$$f_1(x_{(1)}) = n\sum_{k=0}^{\infty}\frac{\xi_k}{x_{(1)}^2} e^{-\frac{\beta(k+1)}{x_{(1)}}}\left[1-\frac{1}{1-e^{-\lambda}} + \sum_{k=0}^{\infty}\upsilon_k e^{-\frac{\beta k}{x_{(1)}}}\right]^{n-1} \quad ; x_{(1)} > 0$$

4. Maximum likelihood estimation (MLE)

Let, $x_1, x_2, ..., x_n$ is a random sample from HLEE with parameters (α, β, λ), then log likelihood function of $\ell(\alpha, \beta, \lambda)$ parameters is given by;

$$\ell n(x;\alpha,\beta,\lambda) = n\ell n(\alpha\beta\lambda) - 2\sum_{j=1}^{n}\ell n(x_j) - n\ell n(1-e^{-\lambda}) - \beta\sum_{j=1}^{n}\frac{1}{x_j}$$
$$+(\alpha-1)\sum_{j=1}^{n}\ell n\left(1-e^{-\frac{\beta}{x_j}}\right) - \lambda\sum_{j=1}^{n}\left\{1-\left(1-e^{-\frac{\beta}{x}}\right)^{\alpha}\right\} \tag{4.1}$$

Maximum Likelihood Estimators of parameters can be obtained by partial differentiating (4.1) w.r.to the parameters. Taking $\xi_j = e^{-\frac{\beta}{x_j}}$, we get,

$$\frac{\partial \ell n(x)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{j=1}^{n}(1-\xi_j) + \lambda\sum_{j=1}^{n}\left\{(1-\xi_j)^{\alpha}\ell n(1-\xi_j)\right\} \tag{4.2}$$

$$\frac{\partial \ell n(x)}{\partial \beta} = \frac{n}{\beta} - \sum_{j=1}^{n}\frac{1}{x_j} + (\alpha-1)\sum_{j=1}^{n}\frac{\xi_j}{x_j(1-\xi_j)} + \sum_{j=1}^{n}\left\{\frac{\alpha}{x_j}\xi_j(1-\xi_j)^{\alpha-1}\right\} \tag{4.3}$$

$$\frac{\partial \ell n(x)}{\partial \lambda} = \frac{n}{\lambda} - \frac{ne^{-\lambda}}{1-e^{-\lambda}} - \sum_{j=1}^{n}\left\{1-\left(1-\xi_j\right)^{\alpha}\right\} \tag{4.4}$$

Solving non-linear equations $\frac{\partial \ln(\ell)}{\partial \alpha} = 0, \frac{\partial \ln(\ell)}{\partial \beta} = 0$ and $\frac{\partial \ln(\ell)}{\partial \lambda} = 0$ for α , β and λ ; we get the maximum likelihood estimate of $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ of the parameters α , β and λ from equation (4.1) directly by using the *optim ()* function of R software R core team (2020).

4.1.Asymptotic Distribution

In order to have approximate confidence interval (CLs) of parameters (α, β, λ), we obtained observed information matrix $O(\hat{\delta})$, which is

$$O(\hat{\delta}) = -\begin{pmatrix} \frac{\partial^2 \ell}{\partial \alpha^2} & \frac{\partial^2 \ell}{\partial \alpha\, \partial \beta} & \frac{\partial^2 \ell}{\partial \alpha\, \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \alpha\, \partial \beta} & \frac{\partial^2 \ell}{\partial \beta^2} & \frac{\partial^2 \ell}{\partial \beta\, \partial \lambda} \\ \frac{\partial^2 \ell}{\partial \alpha\, \partial \lambda} & \frac{\partial^2 \ell}{\partial \beta\, \partial \lambda} & \frac{\partial^2 \ell}{\partial \lambda^2} \end{pmatrix} \tag{4.5}$$

Where, element of Observed Information Matrix are in **Appendix-A.** The Newton Raphson algorithm to maximize the likelihood produces the Observed Information Matrix. From the Observed Information Matrix, we have to introduce Hessian matrix. The inverse of Hessian matrix $\left(-H(\delta)\big|_{\delta=\hat{\delta}}\right)^{-1}$ is called variance-covariance matrix.

Hence, from the asymptotic normality of MLE's, approximate $100(1-\gamma)$ % confidence interval for α, β and λ can be constructed as

$$\hat{\alpha} \pm z_{\gamma/2}\sqrt{\text{var}(\hat{\alpha})} \;,\; \hat{\beta} \pm z_{\gamma/2}\sqrt{\text{var}(\hat{\beta})} \text{ and } \hat{\lambda} \pm z_{\gamma/2}\sqrt{\text{var}(\hat{\lambda})} \,,$$

where, $z_{\gamma/2}$ is the upper percentile of standard normal distribution and the value of variance is obtained from diagonal element of variance-covariance matrix.

5. Data analysis

Simulation modeling solves real-world problems safely and efficiently. Therefore Simulation has been done from equation (3.6) as initial value $\alpha = 2.5, \beta = 3.0$ and $\lambda = 1.0$. Hundredand eleven samples have been drawn and are presented as follows:

(3.16, 2.62, 1.08, 2.72, 3.95, 1.80, 2.03, 5.85, 6.57, 2.27, 0.76, 3.42, 1.66, 1.44, 1.24, 1.13, 1.51, 5.86, 1.17, 1.92, 1.05, 1.03, 2.57, 1.27, 0.91, 0.72, 5.57, 1.92, 0.87, 1.91, 1.83, 3.34, 2.48, 1.92, 2.58, 2.78, 1.42, 1.22, 1.81, 1.16, 3.48, 1.48, 0.76, 6.78, 1.92, 2.58, 2.78, 1.42, 1.22, 1.81, 1.16, 3.48, 1.48, 0.76, 6.78, 1.08, 1.48, 2.13, 2.14, 1.31, 0.92, 5.00, 2.03, 2.76, 5.13, 1.80, 1.40, 0.74, 3.77, 1.74, 1.06, 1.41, 1.06, 1.61, 1.13, 0.87, 0.76, 2.42, 1.22, 0.94, 8.67, 1.49, 1.47, 2.22, 5.05, 1.08, 1.86, 2.58, 1.46, 0.81, 2.16, 0.41, 1.06, 2.36, 3.95, 3.15, 0.91, 1.43, 3.05, 2.43, 2.01, 0.88, 1.88, 1.58, 2.87, 0.60, 1.03, 4.21, 0.85, 8.15, 1.43)

5.1. Parameter Estimation

We have to estimate the value of parameter from equation (4.1) as directly by *optim ()* function of R software, the value of MLE, SE, t-value and p-value was present in table 1.

Table 1: Estimated value, SE, t-value and p-value of parameters

Parameters	MLEs	SE	t- value	p-value
$\hat{\alpha}$	3.2784	0.7841	4.181	<0.001
$\hat{\beta}$	3.2168	0.4760	6.758	<0.001
$\hat{\lambda}$	0.6787	1.1710	0.580	0.562

5.2. Variance covariance matrix

Here, we compared the various criteria to suggest our model. Firstly, we estimated unknown parameters $\delta = (\alpha, \beta, \lambda)$ from MLEs. Furthermore, the variance -covariance matrix of the MLE of proposed model was given as,

$$\begin{array}{c} \\ \alpha \\ \beta \\ \lambda \end{array} \begin{array}{c} \begin{array}{ccc} \alpha & \beta & \lambda \end{array} \\ \begin{pmatrix} 0.61485938 & -0.03450811 & -0.5989780 \\ -0.03450811 & 0.22658132 & 0.4200683 \\ -0.59897796 & 0.42006835 & 1.3713338 \end{pmatrix} \end{array}$$

5.3Model Validation

The total- time-on test (TTT) plot is a graphical procedure to get some idea about the shape of the hazard function. We have plotted the data set in the given figure 4(below left side) and found that the TTT transform is concave, which indicates that the hazard function is unimodal.To check the validity of the model we compute the Kolmogorov- Smironov(KS) distance between the empirical distribution and the fitted distribution function when the parameters are obtained by the method of MLE.We have plotted the empirical distribution function and fitted distribution function in (right side below) the figure and found that the PGIE distribution is suitable for the given data.

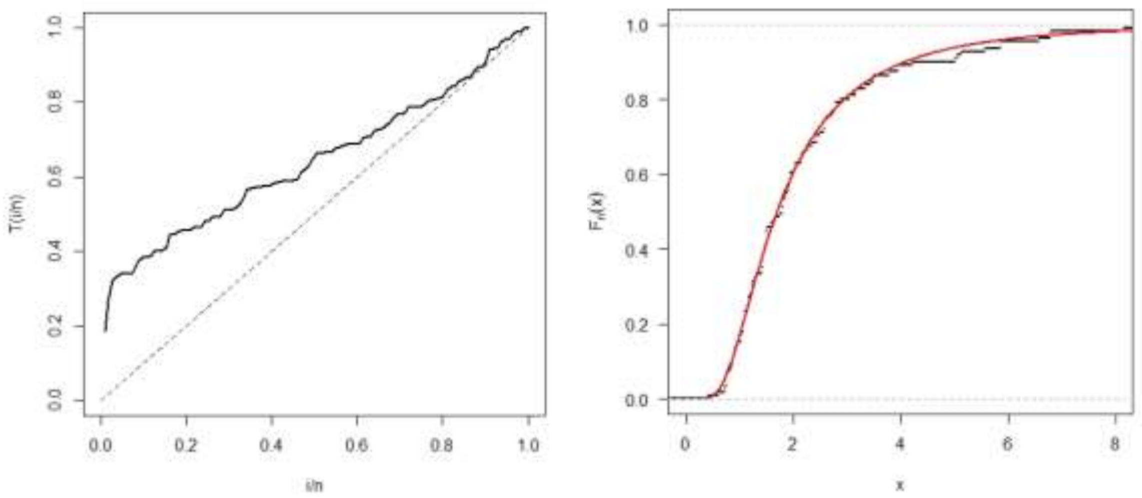

Figure 3 : TTT plot (Left Panel) and estimated CDF vs Empirical distribution function of proposed model (right Panel)

The validation of the finding can also be obtained by inspecting the Probability-Probability (P-P) and quantile – quantile (Q-Q) plots. The P-P plot (left side figure 5) and Q-Q plot (right side figure 5) shows that Poisson generalized inverted Exponential distribution is reasonably good fit to the given data set

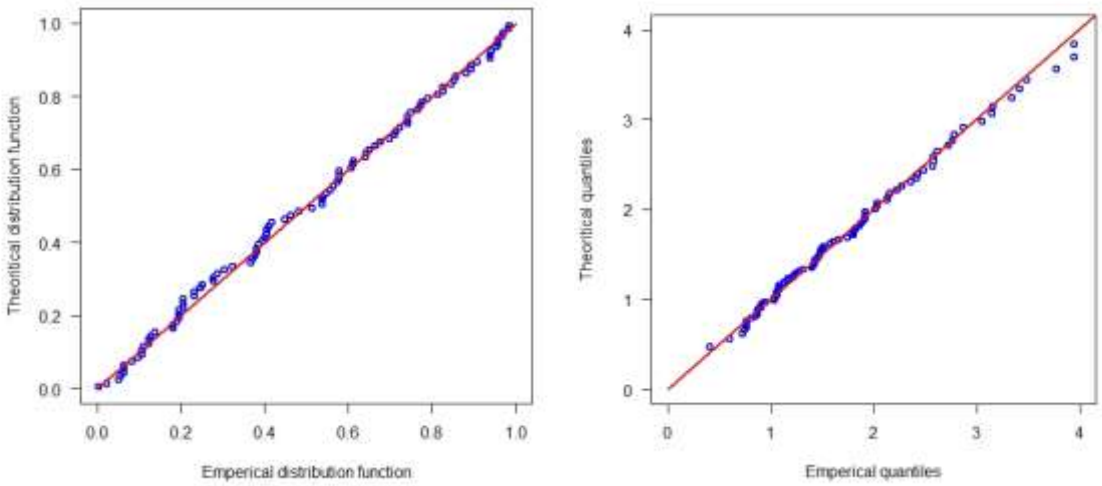

Figure 4: PP plot (left panel), QQplot (right panel) of proposed model fitted by estimated value parameters.

5.4. Model Comparisons

The proposed model is compared with four other competitive models. The competitive models are:

1) A new Poisson Inverted Exponential Distribution by Dhungana, G. P. (2020) having PDF,

$$f(x;\beta,\lambda)=\frac{\beta\lambda}{x^2\left(1-e^{-\lambda}\right)}e^{-\frac{\beta}{x}}e^{-\lambda e^{-\frac{\beta}{x}}};x>0,\beta>0,\lambda>0$$

2) The exponentiated inverted Weibull distribution by Flaih et al.(2012), having PDF,

$$f(x;\beta,\theta)=\beta\theta x^{-(\beta+1)}\left(e^{-x^{-\beta}}\right)^{\theta};x>0,\beta>0,\theta>0$$

3) The Marshall-Olkin logistic-exponential distribution by Mansoor et al, (2019) having PDF

$$f_{MOLE}(x)=\frac{\alpha\theta\lambda e^{\lambda x}(e^{\lambda x}-1)^{-\alpha-1}}{[1+\theta(e^{\lambda x}-1)^{-\alpha}]^2};x>0,\alpha>0,\lambda>0,\theta>0$$

4) A flexible Weibull extension by Bebbington et al. (2007) having PDF

$$f(x)=\left(\alpha+\frac{\beta}{x^2}\right)e^{\left(\alpha x-\frac{\beta}{x}\right)}e^{\left(-e^{\left(\alpha x-\frac{\beta}{x}\right)}\right)};x>0,\alpha>0,\beta>0,\lambda>0$$

Our proposed model PGIE is compared with A new poisson inverted Exponential distribution, The Exponentiated Inverted Weibull distribution, Marshal-olkin logistic exponential distribution and A flexible weibull extension distribution. The value of log-likelihood, AIC(Akaike's information criteria), BIC(Bayesian information criteria) and CIAC(Corrected Akaike's information criteria) are calculated .All these values of the intended model aresmaller in comparisons to other above mentioned distributions. The smaller the value of Likelihood, AIC, BIC and CAIC the better the modelTherefore PGIE distribution is the best fit model (table 2).

Table 2: Estimated value of parameters of different competitive models with AIC, BIC and CIAC

Models	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$	$-\ell(\hat{\theta})$	AIC	BIC	CIAC
PGIE	3.2784	3.2168	0.6787	-	-151.374	308.748	316.8765	308.9723
PIE	-	3.4352	5.1994	-	-169.217	342.435	347.853	342.5461
EIW	-	1.7434	-	1.8346	-169.969	343.939	349.357	344.0501
MOLE	0.6885	-	0.0245	73.1577	-169.643	345.285	353.4145	345.5093
FW	0.2431	1.8807	-		-177.423	358.847	364.265	358.9581

To illustrate the histogram and comparison of density function of PGIE distribution we have taken some well-known distribution for comparisons purpose which are listed above. The Histogram and density function of fitted distribution and selected distributions are presented in figure. The PGIE distribution performs better than any other distributions.

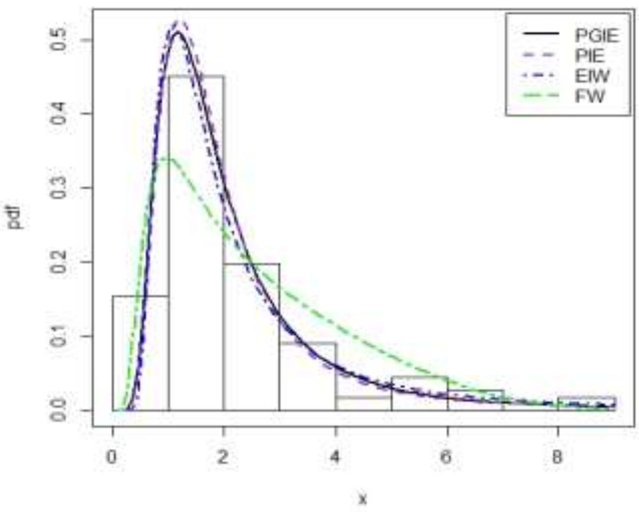

Figure 5: The fitted density function with proposed model with other Competitive models

6 . concluding remark

In this study we have proposed a new family of distributions called Poisson Generalized Inverted exponential (PGIE) distribution. Some distributional properties of the proposed distribution are presented as the shapes of probability density, hazard rate, and survival function. Mean, median, mode, skewness, kurtosis, and Inverse probability weighted moments and order statistics are derived. The hazard function shows the upside curve (concave) shape. The model parameters are estimated by maximum likelihood estimation(MLE). A simulated data set is considered to explore the applicability and suitability of the proposed distribution and found that the proposed model is quite better than other life time models taken in to consideration. The importance and flexibility of the new family is illustrated by different examples mentioned above. We hope that this model may be an alternative reference in the field of survival analysis,probability theory and applied statistics.

Conflict of interest

The authors declare that they have no competing interest.

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Appendix A

$$\frac{\partial^2 \ell n(x)}{\partial \alpha^2} = -\frac{n}{\alpha^2} + \lambda \sum_{j=1}^{n} \left[\left\{ \ell n\left(1-\xi_j\right) \right\}^2 \left(1-\xi_j\right)^{\alpha} \right] \quad (1)$$

$$\frac{\partial^2 \ell n(x)}{\partial \beta^2} = -\frac{n}{\beta^2} - (\alpha-1) \sum_{j=1}^{n} \frac{\xi_j}{\left\{ x_j \left(1-\xi_j\right) \right\}^2} + \lambda \sum_{j=1}^{n} \left\{ \frac{\alpha}{x_j^{\,2}} \xi_j \left(\alpha \xi_j - 1\right) \left(1-\xi_j\right)^{\alpha-2} \right\} \quad (2)$$

$$\frac{\partial^2 \ell n(x)}{\partial \lambda^2} = -\frac{n}{\lambda^2} - n \left[\frac{e^{-\lambda}}{\left(1-e^{-\lambda}\right)^2} \right] \quad (3)$$

$$\frac{\partial^2 \ell n(x)}{\partial \alpha \partial \beta} = \sum_{j=1}^{n} \frac{\xi_j}{x_j \left(1-\xi_j\right)} + \lambda \left[\sum_{j=1}^{n} \left\{ \frac{\xi_j \left(1-\xi_j\right)^{\alpha-1}}{x_j} \left\{ \alpha \ell n\left(1-\xi_j\right) + 1 \right\} \right\} \right] \quad (4)$$

$$\frac{\partial^2 \ell n(x)}{\partial \alpha \partial \lambda} = \sum_{j=1}^{n} \left\{ \left(1-\xi_j\right)^{\alpha} \ell n\left(1-\xi_j\right) \right\} \quad (5)$$

$$\frac{\partial^2 \ell n(x)}{\partial \beta \partial \lambda} = \sum_{j=1}^{n} \left\{ \frac{\alpha}{x_j} \xi_j \left(1-\xi_j\right)^{\alpha-1} \right\} \quad (6)$$

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