SOME NEW HEREDITARY PROPERTIES OF BI-BOUNDED SETS IN AN INJECTIVE TENSOR PRODUCT

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Abstract : Let M and N be two locally convex topological vector spaces over the same field and let $M \otimes N$ denote the algebraic tensor product of M and N. Let us note that $M \otimes N$ is isomorphic to the space $B(M'_\sigma, N'_\sigma)$ of all continuous linear forms on $M'_\sigma \times N'_\sigma$, where M'_σ and N'_σ denote the topological duals of M and N equipped with the weak topologies. In this paper we establish some new hereditary properties of bi-bounded sets in an injective tensor product.

Keywords : hereditary, topological space, tensor product, bi-bounded sets.

INTRODUCTION

A property of a topological space is termed hereditary if and only if every subspace of a space with the property also has the property. Let M and N be two locally convex topological vector spaces over the same field and let $M \otimes N$ denote the algebraic tensor product of M and N.

Let us note that $M \otimes N$ is isomorphic to the space $B(M'_\sigma, N'_\sigma)$ of all continuous linear forms on $M'_\sigma \times N'_\sigma$, where M'_σ and N'_σ denote the topological duals of M and N equipped with the weak topologies.

Definition 1. We define $\in$-topology (or injective topology) on $M \otimes N$ to be the topology carried over from $B(M'_\sigma, N'_\sigma)$ when we regard later as a vector subspace of $\beta \in (M'_\sigma, N'_\sigma)$, the space of separately continuous linear forms on $M'_\sigma \times N'_\sigma$, provided with the topology of uniform convergence on the product of an equi-continuous subset of M' and an equi-continuous subset of N'. The space $M \otimes N$ equipped with the $\in$-topology will be denoted by $M \otimes_\in N$ and is called the injective tensor product of M and N.

Definition 2. We define π-topology (or projective topology) on $M \otimes N$ to be the strongest locally convex topology on $M \otimes N$ for which the canonical bilinear map $(x, y) \to x \otimes y$ of $M \times N$ into $M \otimes N$ is continuous. The space $M \otimes N$ equipped with π-topology is denoted by $M \otimes_\pi N$ and is called the projective tensor product of M and N.

We now establish some new hereditary properties of bi-bounded sets in an injective tensor product.

MAIN RESULTS

Theorem 1. Let P and Q be bounded sets in locally cover spaces M and N respectively. Then $P \circ Q$ is a bounded yet in $M \otimes^\in N$, where

$$P \circ Q = \{x \circ y : x \in P, y \in Q\}\{x, y : P, y \in Q\}$$

φ being the canonical bilinear map of the product vector space $M \times N$ into $M \circ N$.

Proof. Let the locally convex topologies of M and N be defined by the families (p_i) and (q_i) of semi-norms respectively.

Let (U_i) and (V_i) be the corresponding families of zero neighbourhoods in M and N.

Then the semi-norms which define the topology of $M \circ_\in N$ are given by the family $(p_i \circ_\in q_i)$, where

$$(p_i \circ_\in q_i)(z) = \sup_{\substack{f \in u_i \\ g \in v_j}} \left| \sum_{r=1}^{n} < x_r, f >< y_r, g > \right|$$

for $z = \sum_{r=1}^{n} x_r \otimes y_r, x_r \in M, y_r \in N..$

Now let $x \in P$ and $y \in Q$. Then

$$(p_i \otimes q_j)(x \otimes y) = \sup_{\substack{f \in u_i \\ g \in v_i}} |< x, f >< y, g >| \le p_i(x) q_j(y) \quad (1)$$

Now since P and Q are bounded sets in M and N so for each fixed i and j, $p_i\,(P)$ and $q_j\,(Q)$ are bounded sets of real numbers. Hence there exist $t > 0$, $m > 0$ such that

$p_i\,(x \le t$, for all $x \in P$ and $q_j\; v) \le m$ for all $y \in Q$ (2)

Therefore, $(p_i \otimes q_j)(x \otimes y) = t_m$ for all $x \in p$ and $y \in Q$.

Hence every semi-norm $p_i \otimes_\in q_j$ is branded on $P \otimes Q$.

Therefore $P \otimes Q$ is bounded is $M \otimes_\in N$.

Theorem 2. If M and N are (DN)-spices then $M \otimes_\in N$ and $M \otimes_\in N$ are (DN)-spaces. Moreover, the strong topology on $(M \otimes_\in N)'$ coincides with the topology of bi-bounded convergence on $J(M, N)$.

Proof. The topology of bi-bounded convergence on $J(M, N)$ is clearly weaker than the strong topology. To show that they are equal, we must show that the canonical map$J_b\,(M, N) - (M \otimes_\in N)_b$ is continuous

Now since M and N are (DN)-spaces there exist fundamental sequences (P_n) and (Q_n) of bounded subsets of M and N respectively. The topology of bi-bounded convergence on $J(M, N)$ has a base of zero neighbourhoods of the form

$W_{P_n, Q_n} = \{B : |B\ (P_n, Q_n)| \leq 1$ for $n = 1, 2, 3, ...$

Hence this topology is metrizable. Therefore, $J_b(M, N)$ is a bornological space. Thus in view of Kothe (δ28, (3) and (4)), it is sufficient to prove that every sequence converging to zero in $J_b(M, N)$ is bounded in $(M \otimes_{\in} N)'_b$. For this let $z_n \to 0$ in $J_b(M, N)$. Each of the singlet $\{z_1\}, \{z_2\}, \ldots, \{z_n\}$ ... is an equi-continuous subset of $J_b(M, N)$ and (z_n) is bounded in $J_b(M, N)$. So if M and N are (DN)-spaces and z is a bounded subset of $J_b(M, N)$ such that $z = \bigcup_{n=1}^{\infty} z_n$ (where each z_n is $\{z_1, z_2, ...\}$ is equi-continuous subset of $J(M, N)$) then z is equi-continuous. Hence by this we have $\{z_1, z_2, ...\}$ is equi-continuous in $J(M, N)$, so $\{z_1, z_2, ...\}$ is also equi-continuous in $(M \otimes_{\in} N)'$.

Therefore (z_n) is strongly bounded in $(M \otimes_{\in} N)'$.

Hence we have $(M \otimes_{\in} N)'_b = J_b(M, N)$. It follows that $M \otimes_{\in} N$ is a (DN)-space, Hence also $(M \hat{\otimes}_{\in} N)$ is a (DN)-space since $(M \otimes_{\in} N)'_b = (M \otimes_{\in} N)'$.

Corollary. If M and N be strong duals of Banach spaces then the strong dual $(M \otimes_{\in} N)'_b$ of $M \otimes_{\in} N$ is a Banach space on which the strong topology, the norm topology and bi-bounded topology all coincide.

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