Quasi-$g^*\gamma$-open functions in topology

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Abstract

The purpose of this paper is to introduce and study the notions of quasi- $g^*\gamma$-open functions , quasi- $g^*\gamma$-closed functions in topology via $g^*\gamma$-closed

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1. Introduction

In 1997, A. A. El-Atik[5], has introduced and studied the concept of γ-open sets and γ-closed sets in topology. In 2007, E. Ekici[7] has defined and studied the concept of γ-normal spaces in topology and concepts of $g\gamma$-closed sets and γg-closed sets. In [13], [14] and [15], Navalagi et. al. defined and studied the concepts of $g^*\gamma$-closed sets , $g^*\gamma$-open sets, $g^*\gamma$-continuous functions , $g^*\gamma$-irresolute functions , strongly $g^*\gamma$-continuous functions, $g^*\gamma$-open functions, $g^*\gamma$-closed functions and $g^*\gamma$-normal spaces in topology. The aim of this paper is to define and study the concepts of quasi- $g^*\gamma$-open functions , quasi- $g^*\gamma$-closed functions.

2. Priliminaries

In this paper (X,τ) and (Y,σ) (or X and Y) we always mean topological spaces on which no separation axioms are assumed unless otherwise mentioned . For a subset of X, Cl(A) and Int(A) represent the closure of A and the interior of A respectively.

The following definitions and results are useful in the sequel:

Definition 2.1: Let X be a topological space. A subset A is called :

(i)semiopen[8] if $A \subset Cl(Int(A))$,
(ii)preopen[9] if $A \subset Int(Cl(A))$,

(iii)b-open[2] or sp-open[1] or γ-open[5] if $A \subset Cl(Int(A)) \cup Int(Cl(A))$.

The complement of semiopen (resp. peropen, b-open or sp-open or γ-open) set is called semiclosed[4] (resp. preclosed[9], b-closed[2] or sp-closed[1] or γ-closed[5]).

The family of all semiopen (resp. preopen, b-open or sp-open or γ-open) sets of a space X is denoted by SO(X)(resp. PO(X), BO(X), SPO(X) or $\gamma O(X)$).

Definition 2.2: Let A be a subset of a space X, then the intersection of all semi-closed(resp. pre-closed, semipre-closed, γ-closed) sets containing A is called semiclosure[4] (resp. preclosure[6] , semipreclosure[1], γ-closure[5]) of A and is denoted by sCl(A) (resp. pCl(A), spCl(A), $\gamma Cl(A)$).

Definition 2.3: Let A be a subset of a space X, then semi-interior [4](resp. pre-interior[10], semipre-interior[1] , γ-interior[5]) of A is the union of all semiopen(resp. preopen, semipreopem , γ-open) sets contained in A and is denoted by sInt(A) (resp. pInt(A), spInt(A), $\gamma Int(A)$).

Definition 2.4: A subset A of a space X is said to be gγ-closed[7] if $\gamma Cl(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

The complement of gγ-closed set is said to be gγ-open.

Definition 2.5: A subset A of a space X is said to be γg-closed[11] if $\gamma Cl(A) \subset U$ whenever $A \subset U$ and $U \in \gamma O(X)$.

The complement of γg –closed set is said to be γg -open.

The definitions of be gγ-closed set and γg –closed set respectively , defined by E. Ekici[7] and El-Maghrabi[11] are the same.

Definition 2.6: A subset A of a space X is called $g^{*}\gamma$-closed[13] set if $Cl(A) \subset U$ whenever $A \subset U$ and U is γ-open set in X.

Definition 2.7: A subset A of a space X is called $g^{*}\gamma$-open[13] set if $F \subset Int(A)$ whenever $F \subset A$ and F is γ-closed set in X.

The family of all $g^*\gamma$-open sets in topological space X is denoted by $g^*\gamma O(X)$.

Definition 2.8: Let A be a subset of a space X, then the intersection of all $g^*\gamma$-closed sets containing A is called the $g^*\gamma$-closure[13] of A and is denoted by $g^*\gamma Cl(A)$.

Definition 2.9: Let A be a subset of a space X, then the union of all $g^*\gamma$-open sets contained in A is called the $g^*\gamma$-interior[13] of A and is denoted by $g^*\gamma Int(A)$.

Definition 2.10: A set $U\subset X$is said to be $g^*\gamma$-neighbourhood [14](in brief, $g^*\gamma$-nbd) of a point $x\in X$ if and only if there exists $A\in g^*\gamma O(x)$ such that $A\subset U$.

Definition 2.11: A function $f: X\rightarrow Y$ is called semiopen[3](resp. preopen[10], semipreopen[12]), if the image of each open set of X is semiopen(resp. preopen, semopreopen) set in Y.

Definition 2.12: A function $f: X\rightarrow Y$ is called semiclosed[16](resp. preclosed[6], semipreclosed[12,17]), if the image of each open set of X is semiclosed(resp. preclosed, semopreclosed) set in Y.

Definition 2.13: A function $f : X\rightarrow Y$ is said to be $g^*\gamma$-open[13] if the image of open set of X is $g^*\gamma$-open in Y.

Definition 2.14: A function $f : X\rightarrow Y$ is said to be $g^*\gamma$-closed[15] if the image of closed set of X is $g^*\gamma$-closed set in Y.

Definition 2.15: A function $f : X\rightarrow Y$ is said to be $(g^*\gamma,s)$ –open[15](resp. $(g^*\gamma,p)$ –open, $(g^*\gamma,sp)$ –open[15]) if the image of each $g^*\gamma$-open set of X is semiopen(resp. preopen, semipreopen) in Y.

Definition 2.16: A function $f : X\rightarrow Y$ is said to be $(g^*\gamma,s)$ –closed[15](resp. $(g^*\gamma,p)$ –closed, $(g^*\gamma,sp)$ –closed[15]) if the image of each $g^*\gamma$-closed set of X is semiclosed (resp. preclosed, semipreclosed) in Y.

Definition 2.17: A function $f : X\rightarrow Y$ is said to be always $g^*\gamma$-open[13] (resp. always $g^*\gamma$-closed[13]), if the image of each $g^*\gamma$-open(resp. $g^*\gamma$-closed) set of X is $g^*\gamma$-open(resp. $g^*\gamma$-closed) set in Y

Definition 2.18 : A function f: X→ Y is said to be $g^*\gamma$-continuous[13], if the inverse image of each open set of Y is $g^*\gamma$-open set in X.

Definition 2.19 : A function f: X→ Y is said to be $g^*\gamma$-irresolute[13], if the inverse image of each $g^*\gamma$-open set of Y is $g^*\gamma$-open set in X.

Definition 2.20: A space X is said to be $g^*\gamma$-normal[15], if for any pair of disjoint closed sets A and B of X, there exist disjoint $g^*\gamma$-open sets U and V such that $A\subset U$ and $B\subset V$.

3. Quasi-$g^*\gamma$-open functions

We define the following:

Definition 3.1: A function f: X→ Y is said to be quasi-$g^*\gamma$-open if the image of each $g^*\gamma$-open set of X is an open set in Y.

Now we have the following characterizations:

Theorem 3.2: A function f: X→ Y is said to be quasi-$g^*\gamma$-open if and only if for every subset U of X, $f(g^*\gamma Int(U))\subset Int(f(U))$.

Proof: Let f be a quasi-$g^*\gamma$-open function. Now, we have $g^*\gamma Int(U)\subset U$ and $g^*\gamma Int(U)$ is a $g^*\gamma$-open set. Hence we obtain that $f(g^*\gamma Int(U))\subset f(U)$. As $f(g^*\gamma Int(U))$ is open, $f(g^*\gamma Int(U))\subset Int(f(U))$.

Conversely, assume that U is a $g^*\gamma$-open set in X. Then $f(U)=f(g^*\gamma Int(U))\subset Int(f(U))$, but $Int(f(U)\subset f(U)$. Consequently $f(U)=Int(f(U))$, which is open and hence f is quasi- $g^*\gamma$-open function.

Lemma 3.3: If a function f: X→ Y is quasi-$g^*\gamma$-open, then $g^*\gamma Int(f^{-1}(A)) \subset f^{-1}(Int(A))$ for every subset A of Y.

Proof: Let A be any subset of Y. Then, $g^*\gamma Int(f^{-1}(A))$ is a $g^*\gamma$-open set in X and f is quasi-$g^*\gamma$-open, then $f(g^*\gamma Int(f^{-1}(A)))\subset Int(f(f^{-1}(A)))\subset Int(A)$. Thus, $g^*\gamma Int(f^{-1}(A))\subset f^{-1}(Int(A))$.

Theorem 3.4: For a function f: X→ Y, the following are equivalent:

(i) f is quasi-$g^*\gamma$-open

(ii) for each subset U of X, $f(g^*\gamma Int(U))\subset Int(f(U))$

(iii) for each $x \in X$ and each $g^*\gamma$-neighbourhood U of x in X, there exists a neighbourhood V of f(x) in Y such that $V \subset f(U)$.

Proof: (i)$\Rightarrow$(ii) It follows from the theorem 3.2.

(ii)$\Rightarrow$(iii) Let $x \in X$ and U be an arbitrary $g^*\gamma$-neighbourhood of x in X. Then there exist a $g^*\gamma$-open set V in X such that $x \in V \subset U$. Then by (ii), we have f(V)=

$f(g^*\gamma Int(V)) \subset Int(f(V))$ and hence $f(V)=Int(f(V))$. Therefore, it follows that f(V)is open in Y such that $f(x) \in f(V) \subset f(U)$.

(iii)$\Rightarrow$(i) Let U be an arbitrary $g^*\gamma$-open set in X. Then for each $y \in f(U)$, by (iii) there exist a neighbourhoodV_y of y in Y such that $V_y \subset f(U)$. As V_y is a neighbourhood of y, there exist an open set W_y in Y such that $y \in W_y \subset V_y$. Thus

$f(U)=\cup\{W_y: y \in f(U)\}$ which is a open set in Y. This implies that f is quasi-$g^*\gamma$-open function.

Theorem 3.5: A function $f: X \rightarrow Y$ is quasi-$g^*\gamma$-open if and only if for any subset B of Y and for any $g^*\gamma$-closed set F of X containing $f^{-1}(B)$, there exist a closed set G of Y containing B such that $f^{-1}(G) \subset F$.

Proof: Suppose f is quasi-$g^*\gamma$-open function. Let $B \subset Y$ and F be a $g^*\gamma$-closed set of X containing $f^{-1}(B)$. Now, put $G=Y \setminus f(X-F)$. It is clear that $f^{-1}(B) \subset F$ implies $B \subset G$. Since f is quasi-$g^*\gamma$-open , we obtain G as a closed set of Y. Moreover, we have
$f^{-1}(G) \subset F$.

Conversely, let U be a $g^*\gamma$-open set of X and put $B=Y \setminus f(U)$. Then $X \setminus U$ is a $g^*\gamma$-closed set in X containing $f^{-1}(B)$. By hypothesis, there exists a closed set F of Y such that $B \subset F$ and $f^{-1}(F) \subset X \setminus U$. Hence, we obtain $f(U) \subset Y \setminus F$. On the other hand, it follows that $B \subset F$, $Y \setminus F \subset Y \setminus B=f(U)$. Thus, we obtain $f(U)=Y \setminus F$ which is open and hence f is a quasi-$g^*\gamma$-open function.

Theorem 3.6: A function $f: X \rightarrow Y$ is a quasi-$g^*\gamma$-open if and only if $f^{-1}(Cl(B)) \subset$ $g^*\gamma Cl(f^{-1}(B))$ for every subset B of Y.

Proof: Suppose that f is quasi-$g^*\gamma$-open function. For any subset B of Y, $f^{-1}(B) \subset g^*\gamma Cl(f^{-1}(B))$. Therefore, by theorem 3.5, there exists a closed set F in Y such that $B \subset F$ and$f^{-1}(F) \subset g^*\gamma Cl(f^{-1}(B))$. Therefore, we obtain $f^{-1}(Cl(B)) \subset f^{-1}(F) \subset g^*\gamma Cl(f^{-1}(B))$.

Conversely, let $B \subset Y$ and F be a $g^*\gamma$-closed set of X containing $f^{-1}(B)$. Put W=ClY(B), then we have $B \subset W$ and W is closed set and $f^{-1}(W) \subset g^*\gamma Cl(f^{-1}(B)) \subset F$.

Then by theorem 3.5, f is quasi-$g^*\gamma$-open function.

Decompositions of quasi-$g^*\gamma$-open functions:

Theorem 3.7: Let $f: X \to Y$ and $g: Y \to Z$ be two functions. The following statements are valid:

(i) If f is quasi-$g^*\gamma$-open and g is preopen then $g \circ f$ is $(g^*\gamma, p)$ –open function.

(ii) If f is quasi-$g^*\gamma$-open and g is semiopen then $g \circ f$ is $(g^*\gamma, s)$ –open function.

(iii) If f is quasi-$g^*\gamma$-open and g is semipreopen then $g \circ f$ is $(g^*\gamma, sp)$ –open function.

Proof: (i) Let V be any $g^*\gamma$-open set in X. Since f is quasi-$g^*\gamma$-open function, g(V) is open set in Y. Again, g is preopen function and g(V) is open set in Y, then g(f(V))=
$(g \circ f)(V)$ is preopen set in Z. Thus, $g \circ f$ is $(g^*\gamma, p)$ –open function.

(ii) Obvious.

(iii) Obvious.

Theorem 3.8: Let $f: X \to Y$ be $g^*\gamma$-open function and $g: Y \to Z$ be quasi-$g^*\gamma$-open function then $g \circ f$ is open function.

Proof: Obvious.

Theorem 3.9: Let $f: X \to Y$ be quasi-$g^*\gamma$-open function and $g: Y \to Z$ be $g^*\gamma$-open function then $g \circ f$ is always $g^*\gamma$-open function.

Proof: Obvious.

Theorem 3.10: Let $f: X \to Y$ and $g: Y \to Z$ be two functions and $g \circ f: X \to Z$ is quasi- $g^*\gamma$-open function. If g is continuous injective, then f is quasi-$g^*\gamma$-open.

Proof: Let U be a $g^*\gamma$-open set in X. Then $(g \circ f)(U)$ is open in Z, since $g \circ f$ is quasi-$g^*\gamma$-open. Again, g is an injective continuous function, $f(U)= g^{-1}((g \circ f)(U))$ is open in Y. This shows that f is quasi-$g^*\gamma$-open function.

4. Quasi- $g^*\gamma$-closed functions

We define the following:

Definition 4.1: A function $f: X \to Y$ is said to be quasi-$g^*\gamma$-closed if the image of each $g^*\gamma$-closed set of X is closed set in Y.

Now we have the following characterizations:

Lemma 4.2: If a function $f: X \to Y$ is quasi-$g^*\gamma$-closed, then $f^{-1}(Int(B)) \subset g^*\gamma Int(f^{-1}(B))$ for every subset B of Y.

Proof: This proof is similar to the proof of lemma 3.3.

Theorem 4.3: A function $f: X \to Y$ is quasi-$g^*\gamma$-closed if and only if for any subset B of Y and for any $g^*\gamma$-open set G of X containing $f^{-1}(B)$, there exists an open set U of Y containing B such that $f^{-1}(U) \subset G$.

Proof: This proof is similar to the proof of the theorem 3.5.

Theorem 4.4: If $f: X \to Y$ and $g: Y \to Z$ are two quasi-$g^*\gamma$-closed functions, then $g \circ f: X \to Z$ is quasi-$g^*\gamma$-closed function.

Proof: Obvious.

Theorem 4.5: Let X and Y be topological spaces. Then the function $g: X \to Y$ is a quasi-$g^*\gamma$-closed if and only if g(X) is closed in Y and $g(V) \setminus g(X \setminus V)$ is open in g(X) whenever V is $g^*\gamma$-open in X.

Proof: Necessity: Suppose $g: X \to Y$ is a quasi-$g^*\gamma$-closed function. Since X is a $g^*\gamma$-closed, g(X) is closed in Y and $g(V) \setminus g(X \setminus V) = g(V) \cap g(X) \setminus g(X \setminus V)$ is open in g(X) when V is $g^*\gamma$-open in X.

Sufficiency: Suppose g(X) is closed in Y, $g(V) \setminus g(X \setminus V)$ is open in g(X) when V is $g^*\gamma$-open in X and let C be closed in X. Then $g(C)=g(X) \setminus g(X \setminus C) \setminus g(C)$ is closed in g(X) and hence closed in Y.

Corollary 4.6: Let X and Y be topological spaces. Then asurjective function $g: X \to Y$ is quasi-$g^*\gamma$-closed if and only if $g(V) \setminus g(X \setminus V)$ is open in Y whenever V is $g^*\gamma$-open in X.

Proof: Obvious.

Corollary 4.7: Let X and Y be topological spaces and let $g: X \to Y$ be a $g^*\gamma$-continuous quasi-$g^*\gamma$-closed surjection function. Then the topology on Y is $\{g(V) \setminus g(X \setminus V) : V \text{ is } g^*\gamma\text{-open in } X\}$.

Proof: Let W be open in Y. Then $g^{-1}(W)$ is $g^{*}\gamma$-open in X and $g(g^{-1}(W)) \setminus g(X \setminus g^{-1}(W))$ =W. Hence, all open sets of Y are of the form $g(V) \setminus g(X \setminus V)$, V is $g^{*}\gamma$-open set in X. On the other hand, all sets of the form $g(V) \setminus g(X \setminus V)$, V is $g^{*}\gamma$-open in X, are open in Y from corollary 4.6.

Theorem 4.8: Let X and Y be topological spaces with X as $g^{*}\gamma$-normal space. If $g: X \to Y$ is $g^{*}\gamma$-continuous quasi-$g^{*}\gamma$-closed surjection function, then Y is normal. Proof: Let A and B be disjoint closed subsets of Y. Then $g^{-1}(A)$, $g^{-1}(B)$ are disjoint $g^{*}\gamma$-closed subsets of X. Since X is $g^{*}\gamma$-normal, there exists disjoint open sets V and W such that $g^{-1}(A) \subset V$ and $g^{-1}(B) \subset W$. Then $A \subset g(V) \setminus g(X \setminus V)$ and $B \subset g(W) \setminus g(X \setminus W)$. Further by corollary 4.6, $g(V) \setminus g(X \setminus V)$ and $g(W) \setminus g(X \setminus W)$ are open sets in Y and clearly $(g(V) \setminus g(X \setminus V)) \cap (g(W) \setminus g(X \setminus W)) = \phi$. This shows that Y is normal.

Decompositions of quasi-$g^{*}\gamma$-closed functions:

Theorem 4.9: Let $f: X \to Y$ and $g: Y \to Z$ be two functions. Then

(i) If f is $g^{*}\gamma$-closed and g is quasi-$g^{*}\gamma$-closed, then $g \circ f$ is closed.

(ii) If f is quasi-$g^{*}\gamma$-closed and g is $g^{*}\gamma$-closed, then $g \circ f$ is always $g^{*}\gamma$-closed.

(iii) If f is $g^{*}\gamma$-closed and g is quasi-$g^{*}\gamma$-closed, then $g \circ f$ isquasi-$g^{*}\gamma$-closed.

Proof: Obvious.

Theorem 4.9: Let $f: X \to Y$ and $g: Y \to Z$ be two functions such that $g \circ f: X \to Z$ is quasi-$g^{*}\gamma$-closed function.

(i)If f is $g^{*}\gamma$-irresolute surjective, then g is closed.

(ii) If g is $g^{*}\gamma$-continuous injective, then f is always $g^{*}\gamma$-closed.

Proof: (i) Suppose that F is an arbitrary closed set in Y. As f is $g^{*}\gamma$-irresolute, $f^{-1}(F)$ is$g^{*}\gamma$-closed in X. Since $g \circ f$ isquasi-$g^{*}\gamma$-closed and f is surjective, $g \circ f(f^{-1}(F))=$ g(F), which is closed in Z. This implies g is a closed function.

(ii) Suppose F is any $g^{*}\gamma$-closed set in X. Since $g \circ f$ isquasi-$g^{*}\gamma$-closed, $(g \circ f)(F)$ is closedin Z. Again, g is a $g^{*}\gamma$-continuous injective function, $g^{-1}(g \circ f(F))=f(F)$, which is $g^{*}\gamma$-closed in Y. This shows that f is always $g^{*}\gamma$-closed function.

Theorem 4.11: Let $f: X \to Y$ and $g: Y \to Z$ be two functions. Then the following statements are valid:

(i) If f is quasi-$g^{*}\gamma$-closed and g is semiclosed, then $g\circ f$ is $(g^{*}\gamma,s)$ –closed function.

(ii) If f is quasi-$g^{*}\gamma$-closed and g is preclosed then $g\circ f$ is $(g^{*}\gamma,p)$ –closed function.

(iii) If f is quasi-$g^{*}\gamma$-closed and g is semipreclosed then $g\circ f$ is $(g^{*}\gamma,sp)$ –closed function.

Proof: (i) Let V be any $g^{*}\gamma$-closed set in X. Since f is quasi-$g^{*}\gamma$-closed function, g(V) is closed set in Y. Again, g is semiclosed function and g(V) is closed set in Y, then $g(f(V))=(g\circ f)(V)$ is semiclosed set in Z. Thus, $g\circ f$ is $(g^{*}\gamma,s)$ –closed function.

(ii) Obvious.

(iii) Obvious.

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