# LAPLACE TRANSFORM FOR SOLVING SYSTEM OF LINEAR VOLTERRA INTEGROORDINARY DIFFERENTIAL EQUATIONS OF FIRST KIND 

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#### Abstract

The system of Volterra integro-ordinary differential equations generally appears in determining the solutions of heat and mass transfer problem, growth problem of cells, electric circuit problem, drugs delivery problem and spring-mass problem. In this paper, authors present Laplace transform for determining the solution of system of linear Volterra integro-ordinary differential equations of first kind. Two numerical problems have been considered and solved using Laplace transform for explaining the applicability of Laplace transform. Results of numerical problems show that the Laplace transform is very effective for determining the solution of system of linear Volterra integro-ordinary differential equations of first kind.


KEYWORDS: Volterra integro-ordinary differential equation; Laplace transform; Convolution; Inverse Laplace transform
MATHEMATICS SUBJECT CLASSIFICATION: 44A10, 45J05, 45A05, 45D05.
INTRODUCTION: Nowadays, integral transforms are frequently used mathematical techniques for determining the answers of advance problems of space, science, technology and engineering. Providing the exact (analytical) solution of the problem without large calculation work is the most dominant feature of these transforms. Integral transforms were started at least two hundred years ago after the notable works of P.S. Laplace in 1780 and J. Fourier in 1822 among others [1]. Mahgoub [3] defined a new integral transform "Mahgoub Transform. Abdelilah and Hassan [4] introduced the new integral transform "Kamal Transform". Elzaki [5] defined the new integral transform "Elzaki Transform" with its fundamental properties.

The new integral transform "Aboodh Transform" was given by Aboodh [6]. Mohand and Mahgoub [7] introduced the new integral transform "Mohand Transform". Watugula [8] defined a new integral transform namely Sumudu transform and solved differential equations and control engineering problems using it. Maitama and Zhao [9] gave a new integral transform "Shehu transform" for solving differential equations. In 2018, Sadikali [10] introduced a new integral transform: Sadik transform. Mahgoub [11] defined the new integral transform "Sawi Transform". The Upadhyaya integral transform was given by Upadhyaya [12]. Zafar [13] introduced ZZ transform method. Khan and Khan [14] defined Natural transform with its important properties and explain its physical importance by giving applications.

Abdelilah and Hassan [15] used Kamal transform for solving partial differential equations. Aggarwal et al. [16] applied Laplace transform for solving population growth and decay problems. Aggarwal et al. [17] used Laplace transform for solving improper integrals whose integrand consisting error function. Chauhan and Aggarwal [18] applied Laplace transform for determining the solution of convolution type linear Volterra integral equation of second kind. Sharma and Aggarwal [19] used Laplace transform for the solution of Abel's integral equation.

Aggarwal and Sharma [20] determined the solution of first kind linear Volterra integral equation using Laplace transform. Aggarwal and Gupta [21] determined the solution of linear Volterra integro-differential equations of second kind using Kamal transform. Aggarwal and Sharma [22] gave the application of Kamal transform for solving Abel's integral equation. Aggarwal et al. [23] used Kamal transform for solving linear Volterra integral equations. Solution of linear partial integro-differential equations using Kamal transform is given by Gupta et al. [24].

Application of Kamal transform for solving linear Volterra integral equations of first kind was given by Aggarwal et al. [25]. Aggarwal et al. [26] gave the application of Kamal transform for solving population growth and decay problems. Aggarwal [27] defined the Kamal transform of Bessel's functions. Aggarwal and Singh [28] defined Kamal transform of error function. Chauhan and Aggarwal [29] determined the solution of linear partial integro-differential equations using Mahgoub transform. A new application of Mahgoub transform for solving linear ordinary differential equations with variable coefficients was given by Aggarwal et al. [30].

A new application of Mahgoub transform for solving linear Volterra integral equations was given by Aggarwal et al. [31]. Aggarwal et al. [32] determined the solution of linear Volterra integro-differential equations of second kind using Mahgoub transform. Application of Mahgoub transform for solving linear Volterra integral equations of first kind was given by Aggarwal et al. [33]. Application of Mahgoub transform for solving population growth and decay problems was given by Aggarwal et al. [34]. Aggarwal et al. [35] defined Mahgoub transform of Bessel's functions. Mahgoub transform (Laplace-Carson transform) of error function was given by Aggarwal et al. [36].

Gupta [37] solved Abel's integral equation using Mahgoub transform method. Solution of population growth and decay problems by using Mohand transform was given by Aggarwal et al. [38]. Kumar et al. [39] gave the applications of Mohand transform for solving linear Volterra integral equations of first kind. Solution of linear Volterra integral equations of second kind using Mohand transform was given by Aggarwal et al. [40]. Applications of Mohand transform to mechanics and electrical circuit problems were given by Kumar et al. [41].

Aggarwal et al. [42] defined Mohand transform of Bessel's functions. A new application of Mohand transform for handling Abel's integral equation was given by Aggarwal et al. [43]. Aggarwal and others [4449] gave the comparative study of Mohand transform with other integral transforms namely Laplace; Kamal; Elzaki; Aboodh; Sumudu; Mahgoub transforms. Mohand transform of error function was given by Aggarwal et
al. [50]. Aggarwal et al. [51] applied Aboodh transform for solving linear Volterra integro-differential equations of second kind.

Aboodh et al. [52] solved delay differential equations by Aboodh transformation method. Solution of partial integro-differential equations by using Aboodh and double Aboodh transforms methods were given by Aboodh et al. [53]. Mohand et al. [54] determined the solution of ordinary differential equation with variable coefficients using Aboodh transform. A new application of Aboodh transform for solving linear Volterra integral equations was given by Aggarwal et al. [55]. Aggarwal et al. [56] solved the population growth and decay problems by using Aboodh transform method. Aggarwal et al. [57] gave the application of Aboodh transform for solving linear Volterra integral equations of first kind.

Aggarwal and Sharma [58] solved Abel's integral equation by Aboodh transform method. Aggarwal et al. [59] defined the Aboodh transform of Bessel's functions. Aggarwal and Singh [60] determined the Aboodh transform of error function. Aggarwal et al. [61] gave the application of Elzaki transform for solving linear Volterra integral equations of first kind. Application of Elzaki transform for solving population growth and decay problems was given by Aggarwal et al. [62]. Elzaki and Ezaki [63] discussed Elzaki transform and solved ordinary differential equation with variable coefficients using it.

Elzaki and Ezaki [64] gave the applications of new transform 'Elzaki transform'' to partial differential equations. Elzaki transform of Bessel's functions was given by Aggarwal [65]. Shendkar and Jadhav [66] used Elzaki transform and determined the solution of differential equations. Elzaki transform of error function was given by Aggarwal [67]. Aggarwal and Gupta [68] used Sumudu transform for the solution of Abel's integral equation. Aggarwal and Sharma [69] defined Sumudu transform of error function. Application of Shehu transform for handling growth and decay problems was given by Aggarwal et al. [70].

A new application of Shehu transform for handling Volterra integral equations of first kind was given by Aggarwal et al. [71]. Aggarwal and Gupta [72] used Shehu transform for solving Abel's integral equation. Aggarwal and Singh [73] determined the Shehu transform of error function (probability integral). Aggarwal and Sharma [74] gave the Sadik transform of error function (probability integral). Aggarwal and Bhatnagar [75] used Sadik transform for handling population growth and decay problems.

Application of Sadik transform for handling linear Volterra integro-differential equations of second kind was given by Aggarwal et al. [76]. Aggarwal and Bhatnagar [77] solved Abel's integral equation using Sadik transform. Singh and Aggarwal [78] applied Sawi transform for population growth and decay problems. Aggarwal with other scholars [79-86] established the duality relations between different integral transforms. Aggarwal and other scholars [89-98] used different integral transforms and determined the solution of Volterra integro-differential equation of first kind. Primitive of second kind linear Volterra integral equation was determined by Aggarwal et al. [99] using Shehu transform. Higazy et al. [100] used Shehu transformation and determined the number of infected cells and concentration of viral particles in plasma during HIV-1 infections. Higazy et al. [101] applied Sawi decomposition method for Volterra integral equation. The main aim of this
paper is to determine the solution of system of linear Volterra integro-ordinary differential equations of first kind with the help of Laplace transformation.
DEFINITION OF LAPLACE TRANSFORM: The Laplace transform of the function $G(t)$ for all $t \geq 0$ is defined as [2, 87]:
$L\{G(t)\}=\int_{0}^{\infty} G(t) e^{-p t} d t=g(p)$, where $L$ is Laplace transform operator.
TABLE-1 USEFUL PROPERTIES OF LAPLACE TRANSFORM [2, 87]


## TABLE-2 LAPLACE TRANSFORM OF USEFUL FUNCTIONS [87-88]

| S.N. | $G(t)$ | $L\{G(t)\}=g(p)$ |
| :--- | :---: | :---: |
| 1. | 1 | $\frac{1}{p}$ |
| 2. | $t$ | $\frac{1}{p^{2}}$ |
| 3. | $t^{2}$ | $\frac{2!}{p^{3}}$ |
| 4. | $t^{n}, n \in N$ | $\frac{n!}{p^{n+1}}$ |
| 5. | $t^{n}, n>-1$ | $\frac{\Gamma(n+1)}{p^{n+1}}$ |
| 6. | sinat | $\frac{1}{p-a}$ |
| 7. | cosat | $\frac{a}{p^{2}+a^{2}}$ |
| 8. | sinhat | $\frac{p}{p^{2}+a^{2}}$ |
| 9. | coshat | $\frac{a}{p^{2}-a^{2}}$ |
| 10. |  | $\frac{p}{p^{2}-a^{2}}$ |

TABLE 3 INVERSE LAPLACE TRANSFORMS OF USEFUL FUNCTIONS [88]

| S.N. | $g(p)$ | $G(t)=L^{-1}\{g(p)\}$ |
| :--- | :---: | :---: |
| 1. | $\frac{1}{p^{2}}$ | 1 |
| 2. | $\frac{1}{p^{3}}$ | $t$ |
| 3. | $\frac{1}{p^{n+1}}, n \epsilon N$ | $\frac{t^{2}}{2!}$ |
| 4. | $\frac{1}{p^{n+1}, n>-1}$ | $\frac{t^{n}}{n!}$ |
| 5. | $\frac{1}{p-a}$ | $\frac{t^{n}}{\Gamma(n+1)}$ |
| 6. | $\frac{1}{p^{2}+a^{2}}$ | $e^{a t}$ |
| 7. | $\frac{p}{p^{2}+a^{2}}$ | $\frac{\operatorname{sinat}}{a}$ |
| 8. | $\frac{p}{p^{2}-a^{2}}$ | $\cos a t$ |
| 9. |  | $\frac{\operatorname{sinhat}}{a}$ |
| 10. |  | $\operatorname{coshat}$ |

LAPLACE TRANSFORM FOR SOLVING SYSTEM OF LINEAR VOLTERRA INTEGROORDINARY DIFFERENTIAL EQUATIONS OF FIRST KIND: Consider the following general system of convolution type linear Volterra integro-ordinary differential equations of first kind

$$
\begin{align*}
& f_{1}(x)=\left\{\begin{array}{c}
\int_{0}^{x} K_{11}(x-t) u_{1}^{(l)}(t) d t+\int_{0}^{x} K_{12}(x-t) u_{2}(t) d t \\
+\cdots \cdot+\int_{0}^{x} K_{1 n}(x-t) u_{n}(t) d t
\end{array}\right\} \\
& f_{2}(x)=\left\{\begin{array}{c}
\int_{0}^{x} K_{21}(x-t) u_{1}(t) d t+\int_{0}^{x} K_{22}(x-t) u_{2}^{(l)}(t) d t \\
+\cdots \cdot+\int_{0}^{x} K_{2 n}(x-t) u_{n}(t) d t
\end{array}\right\} \tag{1}
\end{align*}
$$

$f_{n}(x)=\left\{\begin{array}{c}\int_{0}^{x} K_{n 1}(x-t) u_{1}(t) d t+\int_{0}^{x} K_{n 2}(x-t) u_{2}(t) d t \\ +\cdots+\int_{0}^{x} K_{n n}(x-t) u_{n}{ }^{(l)}(t) d t\end{array}\right\}$
with $\left\{\begin{array}{c}u_{1}{ }^{(m)}(0)=a_{1 m}, m=0,1,2, \ldots, l-1 ; \\ u_{2}{ }^{(m)}(0)=a_{2 m}, m=0,1,2, \ldots, l-1 ; \\ \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots, \\ u_{n}{ }^{(m)}(0)=a_{n m}, m=0,1,2, \ldots, l-1\end{array}\right\}$
Operating Laplace transform on system (1) and using convolution theorem of Laplace transform, we have

$$
\begin{align*}
L\left\{f_{1}(x)\right\} & =\left[\begin{array}{c}
L\left\{K_{11}(x)\right\} L\left\{u_{1}{ }^{(l)}(x)\right\}+L\left\{K_{12}(x)\right\} L\left\{u_{2}(x)\right\} \\
+\cdots .+L\left\{K_{1 n}(x)\right\} L\left\{u_{n}(x)\right\}
\end{array}\right] \\
L\left\{f_{2}(x)\right\} & =\left[\begin{array}{c}
L\left\{K_{21}(x)\right\} L\left\{u_{1}(x)\right\}+L\left\{K_{22}(x)\right\} L\left\{u_{2}(l)(x)\right\} \\
+\cdots .+L\left\{K_{2 n}(x)\right\} L\left\{u_{n}(x)\right\}
\end{array}\right] \tag{3}
\end{align*}
$$

Using the property "Laplace transforms of derivatives" on system (3), we have

$$
\begin{aligned}
& \left.L\left\{f_{n}(x)\right\}=\left[\begin{array}{c}
L\left\{K_{n 1}(x)\right\} L\left\{u_{1}(x)\right\} \\
+L\left\{K_{n 2}(x)\right\} L\left\{u_{2}(x)\right\} \\
+\cdots+L\left\{K_{n n}(x)\right\}\left\{\begin{array}{c}
p^{l} L\left\{u_{n}(x)\right\} \\
-p^{l-1} u_{n}(0) \\
-p^{l-2} u_{n}^{\prime}(0) \\
-\cdots . . u_{n}^{(l-1)}(0)
\end{array}\right\}
\end{array}\right\}\right]
\end{aligned}
$$

Using equation (2) in system (4), we get

$$
\begin{align*}
& L\left\{f_{1}(x)\right\}=\left[\begin{array}{c}
L\left\{K_{11}(x)\right\}\left\{\begin{array}{c}
p^{l} L\left\{u_{1}(x)\right\} \\
-p^{l-1} a_{10} \\
-p^{l-2} a_{11} \\
-\cdots . .-a_{1(l-1)}
\end{array}\right\} \\
+L\left\{K_{12}(x)\right\} L\left\{u_{2}(x)\right\} \\
+\cdots .+L\left\{K_{1 n}(x)\right\} L\left\{u_{n}(x)\right\}
\end{array}\right] \\
& L\left\{f_{2}(x)\right\}=\left[\begin{array}{c}
L\left\{K_{21}(x)\right\} L\left\{u_{1}(x)\right\} \\
+L\left\{K_{22}(x)\right\}\left\{\begin{array}{c}
p^{l} L\left\{u_{2}(x)\right\} \\
-p^{l-1} a_{20} \\
-p^{l-2} a_{21} \\
-\cdots . .-a_{2(l-1)}
\end{array}\right\} \\
+\cdots . .+L\left\{K_{2 n}(x)\right\} L\left\{u_{n}(x)\right\}
\end{array}\right] \tag{5}
\end{align*}
$$

$$
\left.L\left\{f_{n}(x)\right\}=\left[\begin{array}{c}
L\left\{K_{n 1}(x)\right\} L\left\{u_{1}(x)\right\} \\
+L\left\{K_{n 2}(x)\right\} L\left\{u_{2}(x)\right\} \\
+\cdots .+L\left\{K_{n n}(x)\right\}\left\{\begin{array}{c}
p^{l} L\left\{u_{n}(x)\right\} \\
-p^{l-1} a_{n 0} \\
-p^{l-2} a_{n 1} \\
-\cdots a_{n(l-1)}
\end{array}\right\}
\end{array}\right]\right]
$$

After simplification system (5), we have

$$
\begin{align*}
& {\left[\begin{array}{c}
\left(p^{l} L\left\{K_{11}(x)\right\}\right) L\left\{u_{1}(x)\right\} \\
+L\left\{K_{12}(x)\right\}\left\{u_{2}(x)\right\} \\
+ \\
\ldots \ldots+L\left\{K_{1 n}(x)\right\}\left\{u_{n}(x)\right\}
\end{array}\right]=\left[\begin{array}{c}
L\left\{f_{1}(x)\right\} \\
+\left\{\begin{array}{c}
p^{l-1} a_{10}+p^{l-2} a_{11} \\
+\cdots \\
+a_{1(l-1)}
\end{array}\right\} L\left\{K_{11}(x)\right\}
\end{array}\right]} \\
& {\left[\begin{array}{c}
L\left\{K_{21}(x)\right\}\left\{u_{1}(x)\right\} \\
+\left(p^{l} L\left\{K_{22}(x)\right\}\right) L\left\{u_{2}(x)\right\} \\
+ \\
\ldots \ldots+L\left\{K_{2 n}(x)\right\}\left\{u_{n}(x)\right\}
\end{array}\right]=\left[\begin{array}{c}
L\left\{f_{2}(x)\right\} \\
\left.+\left\{\begin{array}{c}
p^{l-1} a_{20}+p^{l-2} a_{21} \\
+\cdots \cdots \\
+a_{2(l-1)}
\end{array}\right\} L\left\{K_{22}(x)\right\}\right]
\end{array}\right]}  \tag{6}\\
& {\left[\begin{array}{c}
L\left\{K_{n 1}(x)\right\}\left\{u_{1}(x)\right\} \\
+L\left\{K_{n 2}(x)\right\}\left\{u_{2}(x)\right\} \\
+ \\
\ldots \ldots \\
+\left(p^{l} L\left\{K_{n n}(x)\right\}\right) L\left\{u_{n}(x)\right\}
\end{array}\right]=\left[\begin{array}{c}
L\left\{f_{n}(x)\right\} \\
\left.+\left\{\begin{array}{c}
p^{l-1} a_{n 0}+p^{l-2} a_{n 1} \\
+\cdots . . \\
+a_{n(l-1)}
\end{array}\right\} L\left\{K_{n n}(x)\right\}\right]
\end{array}\right]}
\end{align*}
$$

The solution of system (6) is given as


After simplification of above equations, we have the values of $L\left\{u_{1}(x)\right\}, L\left\{u_{2}(x)\right\}, \ldots, L\left\{u_{n}(x)\right\}$. After taking the inverse Laplace transforms on these values, we get the required values of $u_{1}(x), u_{2}(x), \ldots, u_{n}(x)$.

NUMERICAL PROBLEMS: In this part of the paper, some numerical problems have been considered for explaining the complete methodology.
Problem: 1 Consider the following system of convolution type linear Volterra integro-ordinary differential equations of first kind

$$
\left.\begin{array}{c}
\int_{0}^{x} u_{1}{ }^{\prime}(t) d t+\int_{0}^{x} u_{2}(t) d t=2 \sin x \\
\int_{0}^{x} u_{1}(t) d t+\int_{0}^{x} u_{2}{ }^{\prime}(t) d t=0 \tag{7}
\end{array}\right\}
$$

with $u_{1}(0)=0, u_{2}(0)=1$

Operating Laplace transform on system (7) and using convolution theorem of Laplace transform, we have

$$
\begin{gather*}
L\{1\} L\left\{u_{1}{ }^{\prime}(x)\right\}+L\{1\} L\left\{u_{2}(x)\right\}=2 L\{\sin x\}  \tag{9}\\
L\{1\} L\left\{u_{1}(x)\right\}+L\{1\} L\left\{u_{2}{ }^{\prime}(x)\right\}=0
\end{gather*}
$$

Using the property "Laplace transforms of derivatives" on system (9), we have

$$
\left.\begin{array}{c}
\frac{1}{p}\left[p L\left\{u_{1}(x)\right\}-u_{1}(0)\right]+\frac{1}{p} L\left\{u_{2}(x)\right\}=2\left(\frac{1}{p^{2}+1}\right)  \tag{10}\\
\frac{1}{p} L\left\{u_{1}(x)\right\}+\frac{1}{p}\left[p L\left\{u_{2}(x)\right\}-u_{2}(0)\right]=0
\end{array}\right\}
$$

Using equation (9) in system (10), we get

$$
\left.\begin{array}{c}
\frac{1}{p}\left[p L\left\{u_{1}(x)\right\}-0\right]+\frac{1}{p} L\left\{u_{2}(x)\right\}=2\left(\frac{1}{p^{2}+1}\right) \\
\frac{1}{p} L\left\{u_{1}(x)\right\}+\frac{1}{p}\left[p L\left\{u_{2}(x)\right\}-1\right]=0 \tag{11}
\end{array}\right\}
$$

After simplification system (11), we have

$$
\left.\begin{array}{c}
L\left\{u_{1}(x)\right\}+\frac{1}{p} L\left\{u_{2}(x)\right\}=2\left(\frac{1}{p^{2}+1}\right)  \tag{12}\\
\frac{1}{p} L\left\{u_{1}(x)\right\}+L\left\{u_{2}(x)\right\}=\frac{1}{p}
\end{array}\right\}
$$

The solution of system (12) is given by
$L\left\{u_{1}(x)\right\}=\frac{\left|\begin{array}{cc}\left(\frac{2}{p^{2}+1}\right) & \frac{1}{p} \\ \frac{1}{p} & 1\end{array}\right|}{\left|\begin{array}{cc}1 & \frac{1}{p} \\ \frac{1}{p} & 1\end{array}\right|}=\frac{1}{p^{2}+1}$
$\left.L\left\{u_{2}(x)\right\}=\frac{\left|\begin{array}{cc}1 & \left.\frac{2}{p^{2}+1}\right) \\ \frac{1}{p} & \frac{1}{p}\end{array}\right|}{\left|\begin{array}{cc}1 & \frac{1}{p} \\ \frac{1}{p} & 1\end{array}\right|}=\frac{p}{p^{2}+1} \right\rvert\,$

Operating inverse Laplace transforms on system (13), we get the required solution of system (7) with (8) as
$\left.u_{1}(x)=L^{-1}\left\{\frac{1}{p^{2}+1}\right\}=\sin x\right\}$
$\left.u_{2}(x)=L^{-1}\left\{\frac{p}{p^{2}+1}\right\}=\cos x\right\}$.
Problem: 2 Consider the following system of convolution type linear Volterra integro- ordinary differential equations of first kind

$$
\left.\begin{array}{c}
\int_{0}^{x} u_{1}{ }^{\prime}(t) d t-\int_{0}^{x} u_{3}(t) d t=0 \\
\int_{0}^{x} u_{2}^{\prime}(t) d t+\int_{0}^{x} u_{3}(t) d t=0  \tag{14}\\
\int_{0}^{x} u_{1}(t) d t+\int_{0}^{x} u_{2}(t) d t+\int_{0}^{x} u_{3}^{\prime}(t) d t=0
\end{array}\right\}
$$

with $u_{1}(0)=0, u_{2}(0)=1, u_{3}(0)=0$
Operating Laplace transform on system (14) and using convolution theorem of Laplace transform, we have

$$
\left.\begin{array}{c}
L\{1\} L\left\{u_{1}{ }^{\prime}(x)\right\}-L\{1\} L\left\{u_{3}(x)\right\}=0 \\
L\{1\} L\left\{u_{2}{ }^{\prime}(x)\right\}+L\{1\} L\left\{u_{3}(x)\right\}=0 \\
L\{1\} L\left\{u_{1}(x)\right\}+L\{1\} L\left\{u_{2}(x)\right\}+L\{1\} L\left\{u_{3}{ }^{\prime}(x)\right\}=0
\end{array}\right\}
$$

Using the property "Laplace transforms of derivatives" on above system, we have

$$
\left.\begin{array}{c}
\frac{1}{p}\left[p L\left\{u_{1}(x)\right\}-u_{1}(0)\right]-\frac{1}{p} L\left\{u_{3}(x)\right\}=0 \\
\frac{1}{p}\left[p L\left\{u_{2}(x)\right\}-u_{2}(0)\right]+\frac{1}{p} L\left\{u_{3}(x)\right\}=0  \tag{16}\\
\frac{1}{p} L\left\{u_{1}(x)\right\}+\frac{1}{p} L\left\{u_{2}(x)\right\}+\frac{1}{p}\left[p L\left\{u_{3}(x)\right\}-u_{3}(0)\right]=0
\end{array}\right\}
$$

Using equation (15) in system (16), we get

$$
\left.\begin{array}{c}
\frac{1}{p}\left[p L\left\{u_{1}(x)\right\}-0\right]-\frac{1}{p} L\left\{u_{3}(x)\right\}=0  \tag{17}\\
\frac{1}{p}\left[p L\left\{u_{2}(x)\right\}-1\right]+\frac{1}{p} L\left\{u_{3}(x)\right\}=0 \\
\left.l_{1}(x)\right\}+\frac{1}{p} L\left\{u_{2}(x)\right\}+\frac{1}{p}\left[p L\left\{u_{3}(x)\right\}-0\right]=0
\end{array}\right\}
$$

After simplification system (17), we have

$$
\left.\begin{array}{c}
L\left\{u_{1}(x)\right\}-\frac{1}{p} L\left\{u_{3}(x)\right\}=0 \\
L\left\{u_{2}(x)\right\}+\frac{1}{p} L\left\{u_{3}(x)\right\}=\frac{1}{p}  \tag{18}\\
\frac{1}{p} L\left\{u_{1}(x)\right\}+\frac{1}{p} L\left\{u_{2}(x)\right\}+L\left\{u_{3}(x)\right\}=0
\end{array}\right\}
$$

The solution of system (18) is given by
$L\left\{u_{1}(x)\right\}=\frac{\left|\begin{array}{ccc}0 & 0 & -\frac{1}{p} \\ \frac{1}{p} & 1 & \frac{1}{p} \\ 0 & \frac{1}{p} & 1\end{array}\right|}{\left|\begin{array}{ccc}1 & 0 & -\frac{1}{p} \\ 0 & 1 & \frac{1}{p} \\ \frac{1}{p} & \frac{1}{p} & 1\end{array}\right|}=-\frac{1}{p^{3}}$
$L\left\{u_{2}(x)\right\}=\frac{\left|\begin{array}{ccc}1 & 0 & -\frac{1}{p} \\ 0 & \frac{1}{p} & \frac{1}{p} \\ \frac{1}{p} & 0 & 1\end{array}\right|}{\left|\begin{array}{ccc}1 & 0 & -\frac{1}{p} \\ 0 & 1 & \frac{1}{p} \\ \frac{1}{p} & \frac{1}{p} & 1\end{array}\right|}=\frac{1}{p}+\frac{1}{p^{3}}$
$L\left\{u_{3}(x)\right\}=\frac{\left|\begin{array}{ccc}1 & 0 & 0 \\ 0 & 1 & \frac{1}{p} \\ \frac{1}{p} & \frac{1}{p} & 0\end{array}\right|}{\left|\begin{array}{ccc}1 & 0 & -\frac{1}{p} \\ 0 & 1 & \frac{1}{p} \\ \frac{1}{p} & \frac{1}{p} & 1\end{array}\right|}=-\frac{1}{p^{2}}$
Operating inverse Laplace transforms on equations (19), (20) and (21), we get the required solution of system (14) with (15) as
$\left.\begin{array}{c}u_{1}(x)=-L^{-1}\left\{\frac{1}{p^{3}}\right\}=-\frac{x^{2}}{2} \\ \left.u_{2}(x)=L^{-1}\left\{\frac{1}{p}+\frac{1}{p^{3}}\right\}=L^{-1}\left\{\frac{1}{p}\right\}+L^{-1}\left\{\frac{1}{p^{3}}\right\}=1+\frac{x^{2}}{2}\right\} . \\ u_{3}(x)=-L^{-1}\left\{\frac{1}{p^{2}}\right\}=-x\end{array}\right\}$
CONCLUSIONS: In this paper, authors successfully discussed the Laplace transform for the solution of system of linear Volterra integro-ordinary differential equations of first kind and complete methodology explained by considering two numerical problems. The results of numerical problems show that the Laplace transform is very effective and useful integral transform for determining the solution of system of linear Volterra integro-ordinary differential equations of first kind.

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