



## PARAMETER ESTIMATION OF WEIBULL FRECHET DISTRIBUTION VIA BAYESIAN APPROACH

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### Abstract

In this paper, Weibull Frechet distribution is considered for Bayesian analysis. The expressions for Bayes estimators of the parameter have been derived under squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions by using quasi and gamma priors.

### Keywords

Bayesian method, Weibull Frechet distribution, quasi and gamma priors, squared error, precautionary, entropy, K-loss, and Al-Bayyati's loss functions.

### 1. Introduction

Afify, A. Z. et al. [1] introduced the Weibull Frechet distribution. Various of its structural properties including ordinary and incomplete moments, quantile and generating functions, probability weighted moments, Rényi and  $\delta$ -entropies and order statistics were investigated. The probability density function of Weibull Frechet distribution is given by

$$f(x; \theta) = a\delta\lambda^\delta \theta x^{-\delta-1} \exp\left[-a\left(\frac{\lambda}{x}\right)^\delta\right] \left\{1 - \exp\left[-\left(\frac{\lambda}{x}\right)^\delta\right]\right\}^{-a-1} \\ \times \exp\left[-\theta \left\{\exp\left[\left(\frac{\lambda}{x}\right)^\delta\right] - 1\right\}^{-a}\right]; x > 0. \quad (1)$$

The joint density function or likelihood function of (1) is given by

$$f(x; \theta) = (a\delta\lambda^\delta \theta)^n \left(\prod_{i=1}^n x_i^{-\delta-1} e^{-a(\lambda/x_i)^\delta} \left\{1 - e^{-(\lambda/x_i)^\delta}\right\}^{-a-1}\right) \exp\left[-\theta \sum_{i=1}^n \left\{e^{(\lambda/x_i)^\delta} - 1\right\}^{-a}\right] \quad (2)$$

The log likelihood function is given by

$$\log f(x; \theta) = n \log(a\delta\lambda^\delta \theta) + \log\left(\prod_{i=1}^n x_i^{-\delta-1} \exp\left[-a\left(\frac{\lambda}{x_i}\right)^\delta\right] \left\{1 - \exp\left[-\left(\frac{\lambda}{x_i}\right)^\delta\right]\right\}^{-a-1}\right)$$

$$-\theta \sum_{i=1}^n \left\{ \exp \left[ \left( \frac{\lambda}{x_i} \right)^\delta \right] - 1 \right\}^{-a} \quad (3)$$

Differentiating (3) with respect to  $\theta$  and equating to zero, we get the maximum likelihood estimator of  $\theta$  which is given as

$$\hat{\theta} = \frac{n}{\sum_{i=1}^n \left[ \exp \left\{ \left( \lambda/x_i \right)^\delta \right\} - 1 \right]^{-a}}. \quad (4)$$

## 2. Bayesian Method of Estimation

The Bayesian inference procedures have been developed generally under squared error loss function

$$L(\hat{\theta}, \theta) = \left( \hat{\theta} - \theta \right)^2. \quad (5) \quad \text{The Bayes}$$

estimator under the above loss function, say,  $\hat{\theta}_s$  is the posterior mean, i.e.,

$$\hat{\theta}_s = E(\theta). \quad (6)$$

Zellner [2], Basu and Ebrahimi [3] have recognized that the inappropriateness of using symmetric loss function. Norstrom [4] introduced precautionary loss function is given as

$$L(\hat{\theta}, \theta) = \frac{\left( \hat{\theta} - \theta \right)^2}{\hat{\theta}}. \quad (7) \quad \text{The Bayes}$$

estimator under this loss function is denoted by  $\hat{\theta}_p$  and is obtained as  $\hat{\theta}_p = \left[ E(\theta^2) \right]^{1/2}$ .

(8)

Calabria and Pulcini [5] points out that a useful asymmetric loss function is the entropy loss

$$L(\Delta) \propto \left[ \Delta^p - p \log_e(\Delta) - 1 \right]$$

where  $\Delta = \frac{\hat{\theta}}{\theta}$ , and whose minimum occurs at  $\hat{\theta} = \theta$ . Also, the loss function  $L(\Delta)$  has been used in Dey et al. [6] and Dey and Liu [7], in the original form having  $p=1$ . Thus  $L(\Delta)$  can written be as

$$L(\Delta) = b \left[ \Delta - \log_e(\Delta) - 1 \right]; \quad b > 0. \quad (9)$$

The Bayes estimator under entropy loss function is denoted by  $\hat{\theta}_E$  and is obtained by solving the following equation

$$\hat{\theta}_E = \left[ E \left( \frac{1}{\theta} \right) \right]^{-1}. \quad (10)$$

Wasan [8] proposed the K-loss function which is given as

$$L(\hat{\theta}, \theta) = \frac{\left( \hat{\theta} - \theta \right)^2}{\hat{\theta} \theta}. \quad (11) \quad \text{Under K-loss}$$

function the Bayes estimator of  $\theta$  is denoted by  $\hat{\theta}_K$  and is obtained as

$$\hat{\theta}_K = \left[ \frac{E(\theta)}{E(1/\theta)} \right]^{1/2}. \quad (12)$$

Al-Bayyati [9] introduced a new loss function which is given as

$$L\left(\hat{\theta}, \theta\right) = \theta^c \left(\hat{\theta} - \theta\right)^2. \quad (13) \quad \text{Under Al-}$$

Bayyati's loss function the Bayes estimator of  $\theta$  is denoted by  $\hat{\theta}_{Al}$  and is obtained as

$$\hat{\theta}_{Al} = \frac{E\left(\theta^{c+1}\right)}{E\left(\theta^c\right)}. \quad (14)$$

Let us consider two prior distributions of  $\theta$  to obtain the Bayes estimators.

(i) **Quasi-prior:** For the situation where we have no prior information about the parameter  $\theta$ , we may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d}; \theta > 0, d \geq 0, \quad (15) \quad \text{where } d = 0$$

leads to a diffuse prior and  $d = 1$ , a non-informative prior.

(ii) **Gamma prior:** Generally, the gamma density is used as prior distribution of the parameter  $\theta$  given by

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}; \theta > 0. \quad (16)$$

### 3. Posterior density under $g_1(\theta)$

The posterior density of  $\theta$  under  $g_1(\theta)$ , on using (2), is given by

$$\begin{aligned} f(\theta/\underline{x}) &= \frac{\left[ (a\delta\lambda^\delta\theta)^n \left( \prod_{i=1}^n x_i^{-\delta-1} e^{-a(\lambda/x_i)^\delta} \left\{ 1 - e^{-(\lambda/x_i)^\delta} \right\}^{-a-1} \right) \exp \left[ -\theta \sum_{i=1}^n \left\{ e^{(\lambda/x_i)^\delta} - 1 \right\}^{-a} \right] \theta^{-d} \right]}{\int_0^\infty \left[ (a\delta\lambda^\delta\theta)^n \left( \prod_{i=1}^n x_i^{-\delta-1} e^{-a(\lambda/x_i)^\delta} \left\{ 1 - e^{-(\lambda/x_i)^\delta} \right\}^{-a-1} \right) \exp \left[ -\theta \sum_{i=1}^n \left\{ e^{(\lambda/x_i)^\delta} - 1 \right\}^{-a} \right] \theta^{-d} \right] d\theta} \\ &= \frac{\theta^{n-d} \exp \left[ -\theta \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right]}{\int_0^\infty \theta^{n-d} \exp \left[ -\theta \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right] d\theta} \\ &= \frac{\left( \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{n-d+1}}{\Gamma(n-d+1)} \theta^{n-d} e^{-\theta \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a}} \end{aligned} \quad (17)$$

**Theorem 1.** On using (17), we have

$$E(\theta^c) = \frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)} \left( \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{-c}. \quad (18)$$

**Proof.** By definition,

$$\begin{aligned} E(\theta^c) &= \int \theta^c f(\theta/\underline{x}) d\theta \\ &= \frac{\left( \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{n-d+1}}{\Gamma(n-d+1)} \int_0^\infty \theta^{(n-d+c)} e^{-\theta \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a}} d\theta \\ &= \frac{\left( \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{n-d+1}}{\Gamma(n-d+1)} \times \frac{\Gamma(n-d+c+1)}{\left( \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{n-d+c+1}} \end{aligned}$$

$$= \frac{\Gamma(n-d+c+1)}{\Gamma(n-d+1)} \left( \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-c}.$$

From equation (18), for  $c=1$ , we have

$$E(\theta) = (n-d+1) \left( \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-1}. \quad (19)$$

From equation (18), for  $c=2$ , we have

$$E(\theta^2) = [(n-d+2)(n-d+1)] \left[ \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right]^{-2}. \quad (20)$$

From equation (18), for  $c=-1$ , we have

$$E\left(\frac{1}{\theta}\right) = \frac{1}{(n-d)} \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a}. \quad (21)$$

From equation (18), for  $c=c+1$ , we have

$$E(\theta^{c+1}) = \frac{\Gamma(n-d+c+2)}{\Gamma(n-d+1)} \left( \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-(c+1)}. \quad (22)$$

#### 4. Bayes estimators under $g_1(\theta)$

From equation (6), on using (19), the Bayes estimator of  $\theta$  under squared error loss function is given by

$$\hat{\theta}_S = (n-d+1) \left( \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-1}. \quad (23)$$

From equation (8), on using (20), the Bayes estimator of  $\theta$  under precautionary loss function is obtained as

$$\hat{\theta}_P = [(n-d+2)(n-d+1)]^{\frac{1}{2}} \left( \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-1}. \quad (24)$$

From equation (10), on using (21), the Bayes estimator of  $\theta$  under entropy loss function is given by

$$\hat{\theta}_E = (n-d) \left( \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-1}. \quad (25)$$

From equation (12), on using (19) and (21), the Bayes estimator of  $\theta$  under K-loss function is given by

$$\hat{\theta}_K = [(n-d+1)(n-d)]^{\frac{1}{2}} \left( \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-1}. \quad (26)$$

From equation (14), on using (18) and (22), the Bayes estimator of  $\theta$  under Al-Bayyati's loss function comes out to be

$$\hat{\theta}_{Al} = (n-d+c+1) \left( \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-1}. \quad (27)$$

#### 5. Posterior density under $g_2(\theta)$

Under  $g_2(\theta)$ , the posterior density of  $\theta$ , using equation (2), is obtained as

$$f(\theta/x) = \frac{\left[ (a\delta\lambda^\delta\theta)^n \left( \prod_{i=1}^n x_i^{-\delta-1} e^{-a\left(\frac{\lambda}{x_i}\right)^\delta} \left\{ 1 - e^{-\left(\frac{\lambda}{x_i}\right)^\delta} \right\}^{-a-1} \right) e^{-\theta \sum_{i=1}^n \left\{ e^{(\lambda/x_i)^\delta} - 1 \right\}^{-a}} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \right]}{\int_0^\infty \left[ (a\delta\lambda^\delta\theta)^n \left( \prod_{i=1}^n x_i^{-\delta-1} e^{-a\left(\frac{\lambda}{x_i}\right)^\delta} \left\{ 1 - e^{-\left(\frac{\lambda}{x_i}\right)^\delta} \right\}^{-a-1} \right) e^{-\theta \sum_{i=1}^n \left\{ e^{(\lambda/x_i)^\delta} - 1 \right\}^{-a}} \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} \right] d\theta}$$

$$\begin{aligned}
& \frac{\theta^{n+\alpha-1} \exp \left[ - \left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right) \theta \right]}{\int_0^\infty \theta^{n+\alpha-1} \exp \left[ - \left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right) \theta \right] d\theta} \\
&= \frac{\theta^{n+\alpha-1} \exp \left[ - \left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right) \theta \right]}{\Gamma(n+\alpha) \left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{n+\alpha}} \\
&= \frac{\left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{n+\alpha}}{\Gamma(n+\alpha)} \theta^{n+\alpha-1} e^{-\left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right) \theta} \quad (28)
\end{aligned}$$

**Theorem 2.** On using (28), we have

$$E(\theta^c) = \frac{\Gamma(n+\alpha+c)}{\Gamma(n+\alpha)} \left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{-c} \quad (29)$$

**Proof.** By definition,

$$\begin{aligned}
E(\theta^c) &= \int \theta^c f(\theta/x) d\theta \\
&= \frac{\left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{n+\alpha}}{\Gamma(n+\alpha)} \int_0^\infty \theta^{n+\alpha+c-1} e^{-\left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right) \theta} d\theta \\
&= \frac{\left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{n+\alpha}}{\Gamma(n+\alpha)} \frac{\Gamma(n+\alpha+c)}{\left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{n+\alpha+c}} \\
&= \frac{\Gamma(n+\alpha+c)}{\Gamma(n+\alpha)} \left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{-c}.
\end{aligned}$$

From equation (29), for  $c=1$ , we have

$$E(\theta) = (n+\alpha) \left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{-1} \quad (30)$$

From equation (29), for  $c=2$ , we have

$$E(\theta^2) = [(n+\alpha+1)(n+\alpha)] \left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{-2} \quad (31)$$

From equation (29), for  $c=-1$ , we have

$$E\left(\frac{1}{\theta}\right) = \frac{1}{(n+\alpha-1)} \left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right). \quad (32)$$

From equation (29), for  $c=c+1$ , we have

$$E(\theta^{c+1}) = \frac{\Gamma(n+\alpha+c+1)}{\Gamma(n+\alpha)} \left( \beta + \sum_{i=1}^n \left[ \exp \left\{ (\lambda/x_i)^\delta \right\} - 1 \right]^{-a} \right)^{-(c+1)} \quad (33)$$

## 6. Bayes estimators under $g_2(\theta)$

From equation (6), on using (30), the Bayes estimator of  $\theta$  under squared error loss function is given by

$$\hat{\theta}_S = (n + \alpha) \left( \beta + \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-1}. \quad (34)$$

From equation (8), on using (31), the Bayes estimator of  $\theta$  under precautionary loss function is obtained as

$$\hat{\theta}_P = \left[ (n + \alpha + 1)(n + \alpha) \right]^{\frac{1}{2}} \left( \beta + \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-1}. \quad (35)$$

From equation (10), on using (32), the Bayes estimator of  $\theta$  under entropy loss function is given by

$$\hat{\theta}_E = (n + \alpha + 1) \left( \beta + \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-1}. \quad (36)$$

From equation (12), on using (30) and (32), the Bayes estimator of  $\theta$  under K-loss function is given by

$$\hat{\theta}_K = \left[ (n + \alpha)(n + \alpha - 1) \right]^{\frac{1}{2}} \left( \beta + \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-1}. \quad (37)$$

From equation (14), on using (29) and (33), the Bayes estimator of  $\theta$  under Al-Bayyati's loss function comes out to be

$$\hat{\theta}_{Al} = (n + \alpha + c) \left( \beta + \sum_{i=1}^n \left[ \exp\left\{(\lambda/x_i)^\delta\right\} - 1 \right]^{-a} \right)^{-1}. \quad (38)$$

## Conclusion

In this paper, we have obtained a number of estimators of parameter of Weibull Fréchet distribution. In equation (4) we have obtained the maximum likelihood estimator of the parameter. In equation (23), (24), (25), (26) and (27) we have obtained the Bayes estimators under different loss functions using quasi prior. In equation (34), (35), (36), (37) and (38) we have obtained the Bayes estimators under different loss functions using gamma prior. In the above equations, it is clear that the Bayes estimators depend upon the parameters of the prior distribution.

## References

- [1] Afify, A. Z., Yousof, H. F., Cordeiro, G. M., Ortega, E. M. M. and Nofal, Z. M., (2016): "The Weibull Fréchet distribution and its applications". *Journal of Applied Statistics*, Vol. 43, No. 14, 2608–2626.
- [2] Zellner, A., (1986): "Bayesian estimation and prediction using asymmetric loss functions". *Jour. Amer. Stat. Assoc.*, 91, 446-451.
- [3] Basu, A. P. and Ebrahimi, N., (1991): "Bayesian approach to life testing and reliability estimation using asymmetric loss function". *Jour. Stat. Plann. Infer.*, 29, 21-31.
- [4] Norstrom, J. G., (1996): "The use of precautionary loss functions in Risk Analysis". *IEEE Trans. Reliab.*, 45(3), 400-403.
- [5] Calabria, R., and Pulcini, G. (1994): "Point estimation under asymmetric loss functions for left truncated exponential samples". *Comm. Statist. Theory & Methods*, 25 (3), 585-600.
- [6] D.K. Dey, M. Ghosh and C. Srinivasan (1987): "Simultaneous estimation of parameters under entropy loss". *Jour. Statist. Plan. And infer.*, 347-363.
- [7] D.K. Dey, and Pei-San Liao Liu (1992): "On comparison of estimators in a generalized life Model". *Microelectron. Reliab.* 32 (1/2), 207-221.
- [8] Wasan, M.T., (1970): "Parametric Estimation". New York: Mcgraw-Hill.
- [9] Al-Bayyati, H.N., (2002): "Comparing methods of estimating Weibull failure models using simulation". Ph.D. Thesis, College of Administration and Economics, Baghdad University, Iraq.