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Mathematical Model Applications of First Order Ordinary Differential Equation

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ABSTRACT: The major purpose of this paper is to show the application of first order ordinary differential equation as a mathematical model particularly in describing some biological processes and mixing problems. Application of first order ordinary differential equation in modeling some biological phenomena such as logistic population model and prey-predator interaction for three species in linear food chain system have been analyzed. Furthermore, the application in substance mixing problems in both single and multiple tank systems have been demonstrated. Finally, it is demonstrated that the logistic model is more power full than the exponential model in modeling a population model.

Keywords: mathematical model; logistic model; exponential model

1. Introduction

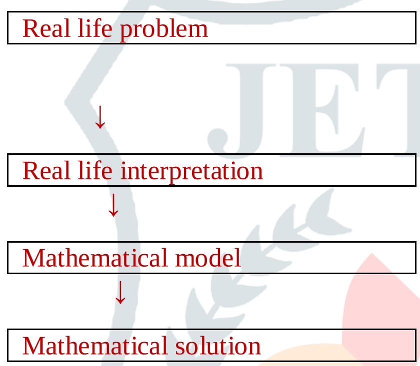
Many real life problems in science and engineering, when formulated mathematically give rise to differential equation. In order to understand the physical behavior of the mathematical representation, it is necessary to have some knowledge about the mathematical character, properties and the solution of the governing differential equation. Many of the principles, or laws, underlying the behavior of the natural world. are statements or relations involving rates at which things happen. When it is expressed in mathematical terms, the relations are equations and the rates are derivatives (Logan, 2017) . If we want to solve a real life problem (usually of a physical nature), we first have to formulate the problem as a mathematical expression in terms of variables, functions, and equations. Such an expression is known as a mathematical model of the given problem. The process of setting up a model, solving it mathematically, and interpreting the result in physical or other term is called mathematical modeling (Bajpai et al., 2018).

Generally a mathematical model is an evolution equation which can potentially describe the evolution of some selected aspects of the real-life problem. The description obtained in solving mathematical problems is generated by the application of the model to the description of real physical behaviors (Bellomo et al., 2007). Since rates of change are represented mathematically by derivatives, mathematical models often involve equations relating an unknown function and one or more derivatives. Such equations are differential equations (Boyce et al.,2017). M any applications, however, require the use of two or more dependent variables, each a function of a single independent variable (typically time) such a problem leads naturally to a system of simultaneous ordinary differential equation (Edwards et al., 2016). The mathematical model to represent a real-life problem is almost always simpler than the actual situation being studied as simplified assumptions are usually required to obtain a mathematical problem that can be solved. Agarap used a mathematical method to represent a simple, hypothetical coin-operated vending machine. A mathematical model of thin film flow with the numerical solution method of solving a third order ordinary differential equations is discussed by (Mechee et al., 2013). Different modeling applications of differential equation are also discussed by [9] - [11].Kolmanovskii, a nd Myshkis (2013) investigated the basic principles of mathematical modeling in applying differential equations on qualitative theory, stability, periodic solutions and optimal control. Cesari (2012) presented the application of differential equations in economics and engineering by examining concrete optimization problems. ksendal (2013) and Simeonov 2007) have analysed the applications of a special type of differential equations called stochastic differential equation and impulsive differential equation, respectively. Frigon and Pouso (2017) dealt with the theory and applications of first-order ordinary differential equations by which the usual derivatives are replaced by Stieltjes derivatives. Scholz and Scholz (2015) discussed the application of first-order ordinary differential equations on Bouguer-Lambert-Beer law in spectroscopy, time constant of sensors, chemical reaction kinetics, radioactive decay, relaxation in nuclear magnetic resonance, and the RC constant of an electrode. In this paper, the application of first order differential equation for modeling population growth or decay, prey predator model, single and multiple tank mixing problems are considered.

2. Preliminaries

2.1 Basic principles and laws of modeling

The process of mathematical modeling can be generalized as



Population law of mass action: The rate of change of a population x (t) due to interaction with a population y(t) is proportional to the product of the populations x (t) and y (t) at a given time t . That is, for a proportionality constant a ,

$$\frac{dx}{dt} = axy \text{-------------} (1)$$

Balance law for population : The net rate of change of the population p(t) is equal to the rate of change of a population in to the ecosystem minus the rate of change of population out of the ecosystem at a time t. That is

$$\frac{dp}{dt} = \left(\frac{dp}{dt}\right)_{in} - \left(\frac{dp}{dt}\right)_{out} \text{------------}(2)$$

First order rate law: The rate at which a population p (t) grows or decays in a first order process is proportional to its population at that time. That is That is, for proportionality constant γ ,

$$\frac{dp}{dt} = \gamma \, p(t) \text{ ---------------}(3)$$

Law of conservation of mass: L et m (t) be the mass of a substance at a time t , then we have

$\frac{dp}{dt}$=0 ----------------(4)

2.2 Linearization of nonlinear system

Definition 1: Linearization is the process of finding the linear approximation of a nonlinear function (system) at a given point. In the study of dynamical systems, linearization is a method for assessing the local stability of an equilibrium point of a system of non-linear differential equations or discrete dynamical system.
Consider a nonlinear system of m first order ordinary differential equations with n variables

$\frac{dxi(t)}{dt}$ = f_i ($x_1,x_2,\ldots\ldots x_n$) , i=1,2,……m ---------------(5)

The Jacobian matrix of the system (5) is the matrix of all first-order partial derivatives of a vector-valued function, f_i ($x_1,x_2,\ldots\ldots x_n$) , i=1,2,……m .It is denoted by J and defined as:

$$J = \frac{\partial(f1,f2,\ldots..fm)}{\partial(x1,x2\ldots\ldots xn)} = \text{Mod} \quad \begin{matrix} \frac{df1}{dx1}\ \frac{df1}{dx2}\ldots\ldots\frac{df1}{dxn} \\ \frac{df2}{dx1}\ \frac{df2}{dx2}\ldots\ldots\frac{df2}{dxn} \\ \ldots\ldots\ldots\ldots\ldots\ldots.. \\ \frac{dfm}{dx1}\ \frac{dfm}{dx2}\ldots\ldots\frac{dfm}{dxm} \end{matrix}$$

Definition 2: We say that the point x0 = ($x_1{}^0$, $x_2{}^0$, ……$x_n{}^0$) is an equilibrium point or fixed point if

$$f_i\ (x_1{}^0,\ x_2{}^0,\ \ldots\ldots x_n{}^0\ ) = 0 \quad \forall\, i$$

The importance of definition 2 lies in the fact that it represents the best linear approximation to a differentiable function near a given point. Based on Jordan and Smith (2007) the linearization form of the non-linear system (5) is given by

$\frac{dx(t)}{dt}$ = Ju(t) ……………..(6)

$$\text{Where } Jf(x_0) = \text{Mod of} \quad \begin{matrix} \frac{d1f(x0)}{dx1}\ \frac{df1(x0)}{dx2}\ldots\ldots\frac{df1(x0)}{dxn} \\ \frac{df2(x0)}{dx1}\ \frac{df2(x0)}{dx2}\ldots\ldots\frac{df2(X0)}{dxn} \\ \ldots\ldots\ldots\ldots\ldots\ldots.. \\ \frac{dfm(x0)}{dx1}\ \frac{dfm(x0)}{dx2}\ldots\ldots\frac{dfm(x0)}{dxm} \end{matrix}$$

$$u(t) = (u_1,\ u_2,\ \ldots\ldots u_n\ )^T$$

$u_1 = (x_1 - x_1^0)$, $u_2 = (x_2 - x_2^0)$ ………… $u_n = (x_n - x_n^0)$

In order to analyze stability of the system, computation of eigenvalues of the corresponding system is the first step.

Definition 3 (Slavik, 2013): The eigenvalues of a tridiagonal matrix A = [a_{ij}] are contained in the union of the intervals[a_{ij}-r_i , a_{ij} +r_i], where

$r_i = \sum_{Mod} a_{ij}$, $1 \le i \le n$

Given an n × n tridiagonal matrix Tn (x)of the form

$$T_n(x) = \det \begin{vmatrix} x & 1 & 0 & 0 \ldots\ldots\ldots 0.0 & 0 \\ 1 & x & 1 & 0 \ldots\ldots\ldots 0..0 & 0 \\ 0 & 1 & x & 1 \ldots\ldots\ldots\ldots 0..0 & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & 0 \ldots\ldots\ldots 1 \;\; x & 1 \\ 0 & 0 & 0 & 0 \ldots\ldots\ldots 0 \;\; 1 & x \end{vmatrix} \qquad (7)$$

and its associated determinant Dn (x) = det |Tn (x) | (Jeffrey, 2010) . Furthermore, t he eigenvalue of Tn (x) is given by $\lambda m = x - 2 \cos(\frac{m\pi}{n+1})$, m= 1,2……n and the eigen vector of Tn(x) is given by $u^m = (u_1^m), (u_2^m), \ldots\ldots\ldots (u_n^m)$

m= 1,2……n

3. Results and Discussion

3.1 Population Growth or Decay Model

Let p (t) denotes the size of population of a country at any time t , then by Balance law for population, we have

$\frac{dp}{dt}$ = B(p,t) - D(p,t) + M (p,t) …………..(8)

where

B(p , t) represents inputs (birth rates),

D(p , t) represents outputs (death rates),

M(p , t) represents net migration.

One of the simplest cases is that assuming a model (8) for birth and death rates are proportional to the population and no migrants. Thus

B(p,t) = b(p,t), D(p,t) = d(p,t) M (p,t) = 0

Hence equation (8) can be reduced to

$\frac{dp}{dt}$ = (b-d) p = γp ……….(9)

where b − d = γ is a proportionality constant which indicates population growth for γ> 0 and population decay

For γ < 0. Since equation (9) is a linear differential equation, we can get a solution of the form:

$p(t) = p_0 e\gamma t$

where p (t0) = 0 is the initial population and ? is called the growth or the decay constant. As a result, the population grows and continues to expand to infinity if γ > 0, while the population will shrink and tend to zero

ifγ < 0. However, populations cannot grow without bound there can be competition for food, resources or space.

Suppose an environment is capable of sustaining no more than a fixed number k of individuals in its population. The quantity k is called the carrying capacity of the environment. Thus, for other models, equation (9) can be expected to decrease as the population p increases in size.

The assumption that the rate at which a population grows (or decreases) is dependent only on the number p (t)present and not on any time-dependent mechanisms such as seasonal phenomena can be stated as

$\frac{dp}{dt}$ = p f(p) ………..(10)

Now, assume that f (p) is linear

f(p) = αp+ β

This is called the logistic population model with growth rate γ and carrying capacity k . Clearly, when assuming p(t) is small compared to k , then the equation reduces to the exponential one which is nonlinear and separable.

The constant solutions p = 0 and p= k are known as equilibrium solution.

3.2 Prey predator model

In this model, we completely characterize the qualitative behavior of a linear three species food chain. Suppose that three different species of animals interact within the same environment or ecosystem. The ecosystem that we wish to model is a l inear three species food chain, where the lowest-level p rey species ? is preyed up on by a mid-level species? , which, in turn, is preyed up on by a top-level predator species ? . Examples of such three species ecosystems include: mouse-snake-owl and worm-robin- falcon (Paullet et al., 2002).The model of predator and prey association includes only natural growthor decay and thepredator-prey interaction

itself. We assume all other relationships (factors) to be negligible. The prey population grows according to a first order rate law in the absence of predators, while the predator population declines according to a first order rate law if the prey population is extinct. If there were no predators in the ecosystem, then the prey's species would, with an added assumption of unlimited food supply, grow at a rate that is proportional to the number of prey species

present at time? (first order rate law):

$\frac{dx}{dt}$ = ax

But when predator species are present, the prey species population is decreased by bxy , b > 0, that is, decreased by the rate at which the preys population are eaten during their encounters with the predator species: adding this

rate to equation gives the model for the prey species population:

$\frac{dx}{dt}$ = ax - bxy

If there were no prey species in the ecosystem, then one might expect that the mid- level species, lacking an adequate food supply, would decline in number according to:

$\frac{dy}{dt}$ = ax - bxy

If there were no prey species in the ecosystem, then one might expect that the mid- level species, lacking an adequate food supply, would decline in number according to:

$\frac{dy}{dt}$ = -cy , c>0

between these two species per unit time is jointly proportional to their populations (the product xy).Thus, when prey species are present, there is a supply of food, so mid-level species are added to the system rate exy , e> 0.

But when top-level predator species are present, the mid-level species population is decreased by gyz , g > 0, decreased by the rate at which the mid-level species population are eaten during their encounters with the toppredator species: Adding this rate to equation gives a model for the mid-level species population:

3.3 Single Tank Mixture Problem Model

Suppose that we have two chemical substances where one is solvable in the other, such as salt and water. Suppose that we have a tank containing a mixture of these substances, and the mixture of them is poured in and the resulting "well-mixed" solution pours out through a valve at the bottom. Now, let's consider Fig. 2 with the following Denotations

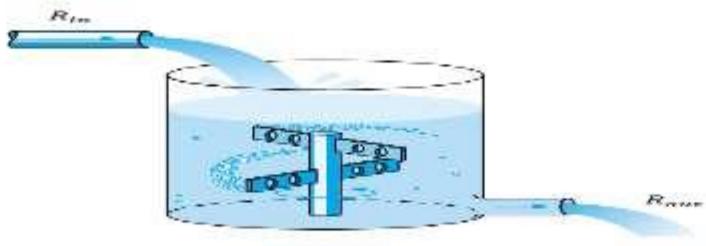


Figure2: mixing Solutions in the tank

C_{in} = Concentration of salt in the solution being poured into the tank

C_{out} = Concentration of salt in the solution being poured out of the tank

R_{in} = rate at which the salt is being poured into the tank

R_{out} = rate at which the salt is being poured out of the tank

There are two types of flow rates for which we can set up differential equations in connection with the mixing problem. Each of these is used to describe what is happening within the tank of liquid. These are volume flow rate and mass flow rate. T he v olume flow rate equation tells us how the amount of liquid in the tank is changing. The net rate of change of the volume in the tank is given by

$$\frac{dy}{dt} = \left(\frac{dy}{dt}\right)_{in} - \left(\frac{dy}{dt}\right)_{out}$$

where v (t) is By the law of conservation of salt, the two rates in the difference represent the constant rate at which liquid is being added to (input flow rate) and at which it is being drained from (output flow rate) the tank. The mass flow rate equation describes the net rate of change of the mass of dissolved substance in the tank.

$$\frac{dm}{dt} = \left(\frac{dm}{dt}\right)_{in} - \left(\frac{dm}{dt}\right)_{out}$$

4. Conclusion

This paper attempted to discuss the application of first order ordinary differential equation i n modeling phenomena of real world problems. The included models are Population growth and decay, Prey-predator interaction, mixing problems in a single tank and multiple tank systems. It is seen that it is possible to represent the population variations of prey and predator relationship to a certain extent of accuracy by mathematical model which is described by systems of non-linear order ordinary differential equations. The logistic model remedies the weakness of exponential model. That is, the exponential model predicts either the population grows without bound or it decays to extinction. But population cannot grow without bound a s t here can be competition for food, resources or space and this effect can be modeled by a logistic model by supposing that the growth rate depends on the population. It is further seen t hat

finding the concentration of the mixed solution after a given period of time leads to the resulting well mixed solution. Finally, this paper believed t hat many problems of future technologies will be solved using ordinary differential equations.

References

(1) Agarap, A.F. (2016). A Mathematical Model of a Hypothetical Coin-Operated Vending Machine Designed Using Concepts in Automata Theory.

(2) Ahsan, Z. (2016). Differential equations and their applications. PHI Learning Pvt. Ltd

(3) Bajpai, A.C., Mustoe, L.R. and Walker, D. (2018). Advanced engineering mathematics.

(4) Bellomo. N, Angelis. E and Delitala. M. (2007). Lecture Notes on Mathematical Modeling in Applied Sciences.

(5) Boyce, W.E., DiPrima, R.C. and Meade, D.B. (2017). Elementary differential equations. John Wiley & Sons.

(6) Cesari, L. (2012). Optimization—theory and applications: problems with ordinary differential equations. Springer Science & Business Media. Vol. 17.

(7) Edwards, C.H., Penney, D.E. and Calvis, D.T. (2016). Differential equations and boundary value problems. Pearson Education Limited.

(8) Elayedi. S. (2005). an introduction to difference equations, springer science and business media Inc., third Edition.

(9) Frigon, M. and Pouso, R.L. (2017). Theory and applications of first-order systems of Stieltjes differential equations. Advances in Nonlinear Analysis, 6(1): 13-36.

(10) Jeffrey. A. (2010). matrix operations for engineers and scientists, e studio calamar s. l. Germany, first Edition.

(11) Jordan .D and Smith. P. (2007). Nonlinear Ordinary Differential Equations, Oxford University Press Inc., Fourth edition.

(12) Kolmanovskii, V. and Myshkis, A. (2013). Introduction to the theory and applications of functional differential equations. Springer Science & Business Media. Vol. 463.

(13) Lambe, C.G. and Tranter, C.J. (2018). Differential equations for engineers and scientists. Courier Dover Publications.

(14) Logan, D. (2017). A first course in differential equations. Springer.

(15) Mechee, M., Senu, N., Ismail, F., Nikouravan, B. and Siri, Z. (2013). A three-stage fifth-order Runge-Kuttamethod for directly solving special third-order differential equation with application to thin film flow problem. Mathematical Problems in Engineering.

(16) Paullet .J, Previte. J and Walls. Z. (2002). lotka - volterra three species food chain, 75: 243