REMARK ON FUZZYRINGS AND IDEALS

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ABSTRACT-Unlike classical logic which requires a deep understanding of a system, exact equations, and precise numeric values, fuzzy logic incorporates an alternative way of thinking, which allows modeling complex systems using a higher level of abstraction originating from our knowledge and experience. The objective of this work is to study the extension of algebraic concepts like rings and ideals to fuzzy settings.

KEY WORDS: Rings, subrings, Ideals, fuzzy rings, fuzzy ideals

I INTRODUCTION:

The term Fuzzy came was initially coined by Zadeh [1] in 1962. In that paper Zadeh called for a mathematics of fuzzy or cloudy quantities which are not describable in terms of probability distributions. This paper lead to the fuzzificationof various algebraic structures. Several mathematicians like Malik and Moderson[2,4], Mukherjee and Sen[3], Sebastian[5], kumar.R[6,7,10], Uckun.M[8], Kaufmann.A[9] worked in this area and have obtained Significant results.

In this paper concepts like Rings, Subrings, Ideals(right & left) and Quotient Ring are extended in their respective fuzzy settings along with certain results related to them are also discussed.

Throughout this work unless otherwise stated R denotes ring(R, +, .), Z, Q,Z_N, ℝ, ℂ denote the ring of Integers, ring of rational numbers, Integer Modulo, Real numbers and Complex numbers.

II BASIC INFORMATION

Fuzzy set introduces unsureness by differentiating the members from nonmembers. Real situations are very often not lucid to understandcan not be described properly. Such situations in which are characterized by imprecision can not be answered just in yes or no. Lotfi A. Zadeh [1] in 1965 introduced the term fuzzy set as an answer to these situations, in which he gave certain grade of membership to each member of a set.The nearer the value of an element to unity, the higher the grade of its membership. This laid the foundation of fuzzy set theory as a generalisation of characteristic function of a set. The membership grades are usually represented by real number values ranging in the closed interval 0 and 1.

It has observedsince the outset of the theory of fuzzy theory that it has wider extent ofapplicability than classical set theory in solving complex situations. Its Applications appear in computer science, artificial intelligence,decision analysis, information science, Washing Machines,expert systems, pattern recognition, management science and operations research. Concept of theoretical mathematics can also befuzzified.

Roughly speaking fuzzy theory has flourishedalong two lines.

(1) As a formal theory which evolved by extending the basic concepts to their Fuzzy settings in the areas such as algebra, topology and so on.

(2) As a very impactful theory which can cope with a large fraction of unpredictable situations of real life and only because of its generality it can be well adapted to different situations.

III FUZZY RINGS AND FUZZY IDEALS

A Fuzzy subset of a non empty set E is characterized by a membership function $\mu : E \longrightarrow [0, 1]$.

If μ, η are fuzzy sets on R then $\mu \subseteq \eta$ if $\mu(x) \leq \eta(x)$ x Є E, Also, their compliment, intersection and union are the fuzzy subsets of R and are defined by $\eta(x) = 1 - \mu(x)$ x Є E, $\mu \cap \eta(x)= \min\{\mu(x), \eta(x)\}$ x Є E and $\mu \cup \eta(x)= \max\{\mu(x), \eta(x)\}$ x Є E. If A = { μ_i : i Є I} be a collection of fuzzy sets where I is a Index set then $\cup\mu_i$and $\cap\mu_i$is Sup μ (x) and Inf μ (x) ∀ x Є R .

3.1**Definition: Fuzzy Ring**- Let R be a Ring, a fuzzy set μ of R is called a Fuzzy Ring of R if

- $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$ for all x, y in R
- $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$ for all x, y in R

3.2**Example**: let μ be a Fuzzy Subset such that $\mu : Z \longrightarrow [0,1]$ defined by

$$\mu(x) = \begin{cases} 0.8 & \text{if } x=0 \\ 0.1 & \text{if } x \neq 0 \end{cases}$$

Then μ is a Fuzzy Ring.

3.3**Proposition:**The characteristic Function ψ_R of a ring R is a fuzzy ring on R

3.4**Proposition:**Let S be a non empty subset of R. Then S is a subring of R iff its charaectisticfunction ψ_s is a fuzzy ring on R

3.5 **Proposition:**A non constant fuzzy set on R is a fuzzyring R iffαA is a subring of R for all αЄIm(A)

IV FUZZY SUBRING

4.1 **Definition: Fuzzy Subring-** Afuzzy subset μ from R to [0, 1] is called a fuzzy Subring of R if for all x, y ЄR the following requirements are met:-

- μ(x-y) ≥ min{ μ(x), μ(y)} for all x, y in R
- μ(xy) ≥ min{ μ(x), μ(y)} for all x, y in R

Example: Define a fuzzy subset μ of R by

$$\begin{cases} t \text{ if x is rational} \\ t' \text{ if x is irrational} \end{cases}$$

where t, ' t Є [0, 1] and t > ' t .

4.2**Definition - "Fuzzy Ideal of a Ring"** from Mukherjee and Sen [3] :A fuzzy subset μ of a ring R to [0, 1] is called as fuzzy ideal of R if it has the following properties:

- μ (x - y) ≥ min {μ (x), μ (y)} x, y Є R.
- μ (xy) ≥ max {μ (x), μ (y)} x, y Є R.

NOTE 1- In case μ is a fuzzy subset from ring R to lattice L then theabove two conditions from Swamy and Swamy[7] will be :

- μ (x - y) ≥μ (x) ∧ μ (y) x, y Є R where ∧ stands for the infimum.
- μ (xy) ≥ μ (x) ∨ μ (y) x, y Є R where ∨ stands for the supremum.

NOTE 2- μ is called fuzzy right ideal if in the above second condition we have μ (xy) ≥ μ (x) and is called as fuzzy left ideal if μ(xy) ≥ μ (y).

4.3**Example**:(i) Consider the ring Z_8 = { 0,1,2,3…..7} with respect to the operations $+_8$ and X_8.

Define Z_8 :μ ⟶[0,1] as

$$\begin{cases} 0.9 \text{ if } x = 0 \\ 0.4 \text{ if } x = 2 \text{ or } 4 \text{ or } 6 \end{cases}$$

0.1 otherwise

(ii) Let R be a non-zero ring. Define fuzzy subsets μ : R → [0, 1] by

$$\mu(x) = \begin{cases} t_0 \text{ if } x = 0 \\ t_3 \text{ otherwise} \end{cases}$$

wheret_i ∈ [0, 1] and $t_0 > t_3$.Here μ_{t_0} , μ_{t_3} be thelevel ideals of μ.

4.4**Proposition** : If μ is a fuzzy subring from ring R to [0, 1], then x, y ∈ R, where, μ (x) ≠ μ (y), μ (x + y) = min {μ (x), μ (y)}.

Proof:- We will prove it by contradiction. Let μ (x + y) > min {μ (x), μ (y)}, without the loss of generality we may take μ (x) < μ (y) therefore μ (x + y)> μ (x). [1] Also, μ (x) ≥ min {μ (x + y), μ (-y)} ⇒ μ (x) ≥ μ (x + y) which is a contradiction to equation [1]. Thus, μ (x + y) ≤ min {μ (x), μ (y)} hence, μ (x + y) = min {μ (x), μ (y)}.

4.5**Proposition**:The intersection of any family of fuzzy subrings of a ring R is again a fuzzy subring of R.

Proof:- Let $\mu_i$be a family of fuzzy subrings of R. To show- μ = ∩ $\mu_i$is a fuzzy subring of R. Consider, μ (x – y) = [∩ μ_i] (x – y) = inf {μ_i(x – y)} ≥ inf {min (μ_i(x), μ_i(y))}

= min {inf (μ_i(x), μ_i(y)} = min {inf μ_i(x), inf μ_i(y)} = min {(∩ μ_i(x)), (∩ μ_i(y))}. Again consider, μ (xy) = [∩ μ_i] (xy) = inf {μ_i(xy)} ≥ inf {min (μ_i(x), μ_i(y))}

= min {inf (μ_i(x), μ_i(y))} = min {infμ_i(x), infμ_i(y)} = min {(∩ μ_i(x)), (∩ μ_i(y))} hence the result.

4.6**Proposition** :Let μ be any fuzzy subring from ring R to [0, 1]. Then the set{x ∈ R : μ (x) = μ (0)} is a subring of R.

Proof:-Part 1- Let x, y belong to the given set. To show- x – y belong to the set.

Consider, μ (x - y) ≥ min {μ (x), μ (y)} = min {μ (0), μ (0)} [since x, y belong to the given set]μ (x - y) ≥ μ (0). Also, μ (0) = μ {(x - y) – (x – y)}≥ min {μ(x – y), μ(x – y)} = μ (x – y). Thus, x – y belong to the set.

Part 2- Let x, y belong to the given set. To show- xy belong to the set. Consider μ (xy) ≥ min {μ (x), μ (y)} = min {μ (0), μ (0)} [since x, y belong to the given set]μ (xy) ≥ μ (0). Also, μ (0) = μ {(xy) – (xy)}≥ min {μ (xy), μ (xy)} = μ (xy). Which implies that xy belong to the set. And thus, the given set a subring of ring R.

4.7**Proposition** :Let μ be any fuzzy subring and θ be any fuzzy ideal from ring R to [0, 1].Then μ ∩ θ is a fuzzy ideal of a subring {x ∈ R : μ(x) = μ(0)}.

Proof:- μ ∩ θ (x-y) = inf μθ (x-y) = inf μ{θ (x-y)} ≥ inf [μ {min (θ (x), θ (y))}]

= inf [min {μ (θ (x), θ (y))}] = min [inf {μθ (x), μθ (y)}] = min [infμθ (x), infμθ (y)]

= min [μ ∩ θ (x), μ ∩ θ (y)].Again considering, μ ∩ θ (xy) = inf {μθ (xy)}

= inf μ {θ (xy)} ≥ inf [μ {max (θ (x), θ (y))}]≥ inf [max {μ (θ (x), θ (y))}]

= max [inf {μθ (x), μθ (y)}]= max [infμθ (x), infμθ (y)] = max [μ ∩ θ (x), μ ∩ θ (y)]. Thus, μ ∩ θ is a fuzzy ideal of the given subring.

4.8**Proposition** : Let μ and η be two fuzzy left ideals from a ring R to [0, 1]. Then μ ∩ η is a fuzzy left ideal of R (similar results hold for right ideals). If μ is a fuzzy right ideal and η a fuzzy left ideal, then μ o η ⊆ μ ∩ η (in the sense that μ o η (x) ≤ μ ∩ η (x) ∀ x ∈ R).

Proof:- Part 1- If μ and η are fuzzy left ideals of a ring R then $\mu \cap \eta$ is a fuzzy left ideal.
Consider, $\mu \cap \eta\,(x - y) = \min\{\mu(x - y), \eta(x - y)\} \geq \min\{\min((\mu(x), \mu(y)), (\min \eta(x), \eta(y))\} \geq \min(\mu \cap \eta(x), \mu \cap \eta(y))$.
Also, $\mu \cap \eta\,(xy) = \min\{\mu(xy), \eta(xy)\} \geq \min\{\mu(y), \eta(y)\}$[since, $\mu(xy) \geq \mu(y)$ and $\eta(xy) \geq \eta(y)$]
$= \mu \cap \eta\,(y)$ therefore, $\mu \cap \eta$ is a fuzzy left ideal.

Part 2- If μ and η are fuzzy right ideals of a ring R then $\mu \cap \eta$ is a fuzzy right ideal.
$\mu \cap \eta\,(x - y) = \min\{\mu(x - y), \eta(x - y)\}$
$\geq \min\{\min(\mu(x), \mu(y)), (\min \eta(x), \eta(y))\} \geq \min(\mu \cap \eta(x), \mu \cap \eta(y))$.
Also, $\mu \cap \eta\,(xy) = \min\{\mu(xy), \eta(xy)\} \geq \min\{\mu(x), \eta(x)\}$ [since, $\mu(xy) \geq \mu(x)$ and $\eta(xy) \geq \eta(x)$] $= \mu \cap \eta\,(x)$.

4.9**Proposition** :If $\mu : R \rightarrow L$ is a fuzzy ideal in R, then $\mu(0) \geq \mu(x)$.

NOTE- Here R stands for ring with multiplicative identity 1and L stands for a completely distributive lattice with least and greatest element in it satisfying the law :- $\vee\{a_i : i \in I\} \wedge \vee\{b_j : j \in J\} = \vee\{a_i \wedge b_j : i \in I, j \in J\}\ \forall\, a_i, b_j \in L$.

Proof:-Consider, $\mu(0) = \mu(0.x) \geq \mu(0) \vee \mu(x) \geq \mu(x)$.

4.10**Proposition** : Let μ be a fuzzy left (right) ideal from ring R to [0, 1]. If $0 \leq t \leq \mu(0)$, then μ_t is a left (right) ideal of R.

Proof:-Part 1-Let μ be a fuzzy left ideal of R.To show- μ_t is a left ideal of R.

Let $0 \leq t \leq \mu(0)$ then $0 \in \mu_t$ implying that $\mu_t \neq \Phi$. Let $x, y \in \mu_t$, since $\mu(x - y) \geq \min\{\mu(x), \mu(y)\} \geq t$ thus, $x - y \in \mu_t$. Again if $r \in R$ and $x \in \mu_t$ then $\mu(rx) \geq \mu(x) \geq t$ thus, $rx \in \mu_t$. Hence μ_t is a left ideal of R.

Part 2- Let μ be a fuzzy right ideal of R. To show- μ_t is a right ideal of R. Let $0 \leq t \leq \mu(0)$ then $0 \in \mu_t$ implying that $\mu_t \neq \Phi$. Let $x, y \in \mu_t$, since $\mu(x - y) \geq \min\{\mu(x), \mu(y)\} \geq t$ thus, $x - y \in \mu_t$. Again if $r \in R$ and $x \in \mu_t$ then $\mu(xr) \geq \mu(x) \geq t$ thus, $xr \in \mu_t$. Hence μ_t is a right ideal of R.

4.11**Proposition** : Let μ be a fuzzy subset from R to [0, 1]. If $\forall\ t \in Im(\mu)$, μ_t is a left(right) ideal of R, then μ is a fuzzy left (right) ideal of R.

Proof:-Part 1- Let μ_t is a left ideal of R, $\forall\ t \in Im(\mu)$. Then $0 \in \mu_t\ \forall\ t \in Im(\mu)$. Hence $\mu(0) \geq t$, $\forall\ t \in Im(\mu)$. Let $x, y \in R$, let $\mu(x) = t$ and $\mu(y) = s$ for some $t, s \in Im(\mu)$. Without loss of generality we may assume that $s \geq t$ then $\mu(y) = s \geq t$ which implies that $x, y \in \mu_t$. Since μ_t is a left ideal of R, $x - y \in \mu_t$. So $\mu(x - y) \geq t = \min\{\mu(x), \mu(y)\}$. Since $y \in \mu_s$ and μ_s is a left ideal of R therefore, $xy \in \mu_s$ for any $x \in R$. Hence $\mu(xy) \geq s = \mu(y)$. Thus, μ is a fuzzy left ideal of R.

Part 2- Let μ_t is a right ideal of R, $\forall\ t \in Im(\mu)$ then $0 \in \mu_t\ \forall\ t \in Im(\mu)$. Hence $\mu(0) \geq t$, $\forall\ t \in Im(\mu)$. Let $x, y \in R$, let $\mu(x) = t$ and $\mu(y) = s$ for some $t, s \in Im(\mu)$. Without loss of generality we may assume that $s \geq t$ then $\mu(y) = s \geq t$ which implies that $x, y \in \mu_t$. Since μ_t is a right ideal of R, $x - y \in \mu_t$. So $\mu(x - y) \geq t = \min\{\mu(x), \mu(y)\}$. Since $x \in \mu_t$ and μ_t is a right ideal of R therefore, $xy \in \mu_t$ for any $y \in R$. Hence $\mu(xy) \geq t = \mu(x)$. Thus, μ is a fuzzy right ideal of R.

V FUZZY COSET

5.1**Definition-"Fuzzy coset"**: Let μ be a fuzzy ideal of R. For any x Є R define a map
$\breve{\mu}_x(r) = \mu(x\text{-}r)$ for all r Є R. Then $\breve{\mu}_x$ is a fuzzy coset of R determined by x and μ

Let μ be a fuzzy ideal of a ring and $\mathcal{T}$ be the set of all fuzzy cosets of A then we have a following theorem: $\mathcal{T}$ is a ring under addition and multiplication$\breve{\mu}_x \oplus \breve{\mu}_y = \breve{\mu}_{x+y}$and $\breve{\mu}_x \odot \breve{\mu}_y = \breve{\mu}_{xy}$for all x, y Є R. Define a map $\bar{A} : \mathcal{T} \longrightarrow [0,1]$ by$\bar{\mu}(\breve{\mu}_x) = \mu(x)$ for all x Є R then $\bar{\mu}$ is a fuzzy subring and fuzzy ideal of $\mathcal{T}$. Also, Letμ be a fuzzy Ideal of R then the map defined above is called as Fuzzy Quotient Ring determined by μ.

5.2**Example**: Let R be a ring Z_{10} and E be a set {0, 2,4,6,8} define a fuzzy subset as follows:

$$\mu(x) = \begin{cases} 1 \text{ if } x \in E \\ \tfrac{1}{2} \text{ if otherwise} \end{cases}$$

then μ is afuzzy ideal also it is easy to show

$$\breve{\mu}_x(r) = \begin{cases} 1 \text{ if } & \begin{cases} x, r \in E \\ x, r \in Z_{10}/E \end{cases} \\ 1/2 \text{ if } & \begin{cases} x \in E,\ r \in Z_{10}/E \\ r \in E,\ x \in Z_{10}/E \end{cases} \end{cases}$$

$$\text{and}\ \bar{\mu}(\breve{\mu}_x) = \begin{cases} 1 \text{ if } x \in E \\ \tfrac{1}{2} \text{ otherwise} \end{cases}$$

5.3**Proposition**: If μ is any fuzzy ideal from a ring R to [0, 1], then the mapf : R $\rightarrow$ R/μ defined by $f(x) = \mu x^*$ for all x Є R, is a homomorphism with kernel μ_t, where $t = \mu(0)$

5.4 **Proposition**:Let μ is a fuzzy ideal from ring R to [0, 1]. Define a mapμ' : R/μ $\longrightarrow$ [0, 1] by $\mu'(\mu x^*) = \mu(x)$ for all x Є R then, μ' so formed is a fuzzy ideal of R/μ.

5.5**Preposition**:If μ is any fuzzy ideal from ring R to [0, 1], then each fuzzy ideal of R/μ corresponds in a natural way to a fuzzy ideal of R

VI CONCLUSION:

During this work we have studied the extension of algebraic concepts like rings, subrings, ideals and fuzzy quotient rings to their fuzzy settings.The Present paper can provide better understanding of the concept to the readers and open new areas of research for them

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