POSITIVE SEMIDEFINITE MATRICES WITH REFERENCE TO INDEFINITE INNER PRODUCT

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ABSTRACT : We discuss about the characterization of positive semidefinite matrices and positive semidefinite block matrices in indefinite inner product space.

KEYWORDS : Indefinite matrix product; Indefinite inner product space; EP matrix; Range symmetric matrix.
MATHEMATICS SUBJECT CLASSIFICATION(2010) : 15A09, 46C20

1. INTRODUCTION

An indefinite inner product in $\mathbb{C}^n$ is a conjugate symmetric sesquilinear form $[x,y]$ together with the regularity condition that $[x,y]=0, \forall y \in \mathbb{C}_J^n$ only when $x=0$. Any indefinite inner product is associated with a unique invertible complex matrix J (called a weight) such that $[x,y]=\langle x,Jy\rangle$, where $\langle\cdot,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{C}^n$. We also make an additional assumption on J, that is, $J^2=I$, to present the results with much algebraic ease.

Investigations of linear maps on indefinite inner product employ the usual multiplication of matrices which is induced by the Euclidean inner product of vectors ([2],[6]). This causes a problem as there are two different values for dot product of vectors. To overcome this difficulty, Kamaraj, Ramanathan and Sivakumar introduced a new matrix product called indefinite matrix multiplication and investigated some of its properties in [6]. More precisely, the indefinite matrix product of two matrices A and B of sizes $m\times n$ and $n\times l$ complex matrices, respectively, is defined to be the matrix $A\circ B=AJ_nB$. The adjoint of A, denoted by $A^{[*]}$ is defined to be the matrix $J_nA^*J_m$, where J_m and J_n are weights.

Many properties of this product are similar to that of the usual matrix product ([6]). Moreover, it not only rectifies the difficulty indicated earlier, but also enables us to recover some interesting results in indefinite inner product spaces in a manner analagous to that of the Euclidean case. Kamaraj, Ramanathan and Sivakumar also established in [6] that in the setting of indefinite inner product spaces, the indefinite matrix product is more appropriate that the usual matrix product. Recall that the Moore-Penrose inverse exists if and only if $rank(AA^*)=rank(A^*A)=rank(A)$. If we take $A=\begin{pmatrix}1&1\\1&1\end{pmatrix}, J=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, then $AA^{[*]}$ and $A^{[*]}A$ are both the zero matrix and so $rank(AA^{[*]})<rank(A)$, thereby proving that the Moore-Penrose inverse doesnot exist with respect to the usual matrix product. However, it can be easily verified that with respect to the indefinite matrix product, $rank\left(A\circ A^{[*]}\right)=rank\left(A^{[*]}\circ A\right)=rank(A)$.Thus, the Moore-Penrose J-inverse with real or complex entries exists over an indefinite inner product, whereas a similar result is false with respect to the usual matrix multiplication. It is therefore really pertinant to extend the study of generalized inverses to the setting of indefinite inner product.

In this paper we study about the positive semidefinite matrices in $\mathscr{I}$. We have also established the characterization theorems in $\mathscr{I}$. Further, we have determined the properties of block matrices over indefinite inner product space. The following notations are used in this paper, $\mathscr{I}$ denotes the indefinite inner product space, $\mathbb{C}^{m\times n}$, $R(A), N(A)$ denotes the class of $m\times n$ matrices, range space, null space in Euclidean space and $\mathbb{C}^{m\times n}_{J_m,J_n}$, $Ra(A), Nu(A)$ denotes the class of $m\times n$ matrices, range space, null space in indefinite inner product space respectively.

2. PRELIMINARIES

We first recall the notion of an indefinite multiplication of matrices.

Definition 2.1. [5]**1** Let $A\in\mathbb{C}^{m\times n}_{J_m,J_n}$, $B\in\mathbb{C}^{n\times k}_{J_n,J_k}$. Let J_n be an arbitrary but fixed $n\times n$ complex matrix such that $J_n=J_n^*=J_n^{-1}$. The indefinite matrix product of A and B (relative to J_n) is defined by $A\circ B=AJ_nB$.

Definition2 **2.2.** [5] For $A\in\mathbb{C}^{m\times n}_{J_m,J_n}$, $A^{[*]}=J_nA^*J_m$ is the adjoint of A relative to J_n and J_m.

Definition3 **2.3.** [5] A matrix $A\in\mathbb{C}^{n\times n}_{J_n}$ is said to be J-invertible if there exists $X\in\mathbb{C}^{n\times n}_{J_n}$, such that $A\circ X=X\circ A=J_n$ such an X is denoted by $A^{[-1]}=JA^{-1}J$.

Remark 2.4. [5] **4** For the identity matrix J, it reduces to a generalized inverse of A and $A_J\{1\}=A\{1\}$. It can be easily verified that X is a generalized inverse of A under the indefinite matrix product if and only if $J_nX\ \ J_m$ is a generalized inverse of A under the usual product of matrices. Hence $A_J\{1\}=\{\ X: J_nX\ \ J_m$ is a generalized inverse of A $\}$.

Definition 2.5. [5]**5** For $A\in\mathbb{C}^{m\times n}_{J_m,J_n}$, $X\in\mathbb{C}^{n\times m}_{J_n,J_m}$ is called the Moore - Penrose J-inverse of A if it satisfies the following equations:(i)$A\circ X\circ A=A$, (ii)$X\circ A\circ X=X$, (iii)$(A\circ X)^{[*]}=A\circ X$, (iv)$(X\circ A)^{[*]}=X\circ A$. Such an X is denoted by $A^{[\dagger]}$ and represented as $A^{[\dagger]}=J_nA^{\dagger}J_m$.

Definition 2.6. [5]**6** The range space of $A\in\mathbb{C}^{m\times n}_{J_m,J_n}$ is defined by $Ra(A)=\{y=A\circ x\in\mathbb{C}^m: x\in\mathbb{C}^n\}$. The null space of $A\in\mathbb{C}^{m\times n}_{J_m,J_n}$ is defined by $Nu(A)=\{x\in\mathbb{C}^n: A\circ x=0\}$.

Property 2.7. [5]7 Let $A \in \mathbb{C}_{J_n}^{m\times n}$. Then

(i)$(A^{[*]})^{[*]} = A$.
(ii)$(A^{[\dagger]})^{[\dagger]} = A$.
(iii)$(AB)^{[*]} = B^{[*]}A^{[*]}$.
(iv)$Ra(A^{[*]}) = Ra(A^{[\dagger]})$.
(v)$Ra(A \circ A^{[*]}) = Ra(A), Ra(A^{[*]} \circ A) = Ra(A^{[*]})$.
(vi)$Nu(A \circ A^{[*]}) = Nu(A^{[*]}), Nu(A^{[*]} \circ A) = Nu(A)$.

Definition 2.8. [5]8 A is range symmetric in $\mathscr{J}$ if and only if $Ra(A) = Ra(A^{[*]})$ (or) equivalently $Nu(A) = Nu(A^{[*]})$.

Remark 2.9. [5]9 In particular for $J = I_n$, this reduces to the definition of range symmetric matrix in unitary space (or) equivalently to an EP matrix.

Theorem 2.10. [5] **10** For $A \in C_{J_n}^{n\times n}$, the following are equivalent:

(i) A is range symmetric in $\mathscr{J}$.
(ii) AJ is EP.
(iii) JA is EP.
(iv) $Nu(A) = Nu(A^{[*]})$.
(v) $A \circ A^{[\dagger]} = A^{[\dagger]} \circ A$.
(vi) $(A^{\dagger}A)^{[*]} = JA^{\dagger}AJ = AA^{\dagger}$.
(vii) A is J-EP.

Theorem11 2.11. [1] Let $A, B \in \mathbb{C}^{n\times n}$ then the following equivalence hold

(i) $R(A) \subseteq R(A^*) \Leftrightarrow N(A) \subseteq N(C) \Leftrightarrow C = CA^{(1)}A$ for every $A^{(1)} \in A\{1\}$
(ii) $R(B) \subseteq R(A) \Leftrightarrow N(A^*) \subseteq N(B^*) \Leftrightarrow B = AA^{(1)}B$ for every $A^{(1)} \in A\{1\}$
(iii) $CA^{(1)}B$ is invariant for every $A^{(1)} \in A\{1\}$

Lemma 122.12. [3] Let A and B be matrices in $\mathscr{J}$ then $N(A^*) \subseteq N(B^*)$ if and only if $Nu(A^{[*]}) \subseteq Nu(B^{[*]})$.

3. CHARACTERIZATION OF POSITIVE SEMIDEFINITE MATRICES

Definition 3.1 13 A matrix $A \in \mathbb{C}^{n\times n}$ is said to be Positive semidefinite(PSD) in $\mathscr{J}$ denoted as $A \geq_I 0 \Leftrightarrow A$ is $J-EP$ and $[Ax, x] \geq 0$, for all $x \in \mathbb{C}^n$.

Theorem 3.214 *For* $A \in \mathbb{C}^{n\times n}, A \geq_I 0 \Leftrightarrow JA \geq 0 \Leftrightarrow AJ \geq 0$.

Proof. Suppose $A \geq_I 0$, then by Definition 3.1, A is $J-EP$ and $[Ax, x] \geq 0$, for all $x \in \mathbb{C}^n$.
From Theorem 2.10 it follows that both JA and AJ are EP.
$A \geq_I 0 \Leftrightarrow [Ax, x] \geq 0$ and A is $J-EP$
$\Leftrightarrow \langle Ax, Jx\rangle \geq 0$ and A is $J-EP$
$\Leftrightarrow \langle JAx, x\rangle \geq 0$ and JA is EP
$\Leftrightarrow JA \geq 0$.
Similarly, $A \geq_I 0 \Leftrightarrow AJ \geq 0$ can be proved. ■

Corollary 3.315 *For* $A \in \mathbb{C}^{n\times n}, A \geq_I 0 \Leftrightarrow A = S^{[*]}JS$, *for some* $S \in \mathbb{C}^{n\times n}$.

Proof. $A \geq_I 0 \Leftrightarrow JA \geq 0 \Leftrightarrow JA = S^*S \Leftrightarrow A = JS^*J^2S \Leftrightarrow A = S^{[*]}JS$. ■

Corollary 3.416 *For* $A \in \mathbb{C}^{n\times n}, A \geq_I 0 \Leftrightarrow A^{[*]} \geq_I 0$

Proof. Since $A \geq_I 0$, by Definition (3.1), A is $J-EP$ and hence $Ra(A) = Ra(A^{[*]})$,
by Theorem 3.4 [4] and Property (2.7) we get $A^{[*]}$ is $J-EP$.
For any $x \in \mathbb{C}^n$, $[A^{[*]}x, x] = [(S^{[*]}JS)^{[*]}x, x] = [x, (S^{[*]}JS)x] = [Sx, JSx] = \langle Sx, Sx\rangle$
$= ||Sx||^2 \geq 0 \Rightarrow [A^{[*]}x, x] \geq 0$. Thus $A \geq_I 0 \Rightarrow A^{[*]} \geq_I 0$, $A^{[*]} \geq_I 0 \Rightarrow (A^{[*]})^{[*]} = A \geq_I 0$ follows from the above steps and using Property (2.7), we have $A^{[*]} \geq_I 0 \Rightarrow A \geq_I 0$. ■

Theorem 3.517 *For* $A \in \mathbb{C}^{n\times n}, A \geq_I 0$ *then for any* P, $P^{[*]}AP \geq_I 0$.

Proof. Since $A \geq_I 0$, by Definition (3.1), $Ra(A) = Ra(A^{[*]})$ and $[Ax, x] \geq 0$, for all $x \in \mathbb{C}^n$.
Let $B = P^{[*]}AP$, $Ra(B^{[*]}) = Ra((P^{[*]}AP)^{[*]}) = Ra(P^{[*]}A^{[*]}P) = Ra(P^{[*]}) = Ra(P^{[*]}AP) = Ra(B)$, which implies B is $J-EP$.
Now, for $x \in \mathbb{C}^n$, $[Bx, x] = [P^{[*]}APx, x] = [APx, Px] = [Ay, y] \geq 0$, where $y = Px$, which implies $B = P^{[*]}AP \geq_I 0$. ■

Theorem 3.618 *For* $A \in \mathbb{C}^{n\times n}, A \geq_I 0 \Leftrightarrow A^{[\dagger]} \geq_I 0$.

Proof. $A \geq_I 0 \Leftrightarrow JA \geq 0$ [By Theorem 3.2]
$\Leftrightarrow (JA)^{[\dagger]} \geq 0$
$\Leftrightarrow A^{[\dagger]}J \geq 0$
$\Leftrightarrow A^{[\dagger]} \geq_I 0$. ■

4. POSITIVE SEMIDEFINITE BLOCK MATRICES IN $\mathscr{I}$.

In this section, we derive equivalent conditions for a block matrix to be positive semidefinite(psd) in $\mathscr{I}$ and establish formulae for $\{1\}$-inverse, $\{1,2\}$-inverse, $\{1,3\}$-inverse and $\{1,4\}$-inverse of positive semidefinite block matrices in $\mathscr{I}$.

Lemma 4.1 [1]**19** *Let* $M = \begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$ *be a psd matrix where* A, D *are Hermitian. Then* $M \geq 0 \Leftrightarrow A \geq 0,\ AA^\dagger B = B$ *and* $D - B^* A^\dagger B \geq 0$.

Theorem 4.220 *Let* $M = \begin{pmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{pmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}$ *then* $M \geq_I 0 \Leftrightarrow A \geq 0,\ X^{[*]}A = (AX)^{[*]},\ Nu(A^{[*]}) \subseteq Nu((AX)^{[*]})$ *and* $X^{[*]}AA^\dagger AX \geq X^{[*]}AX$.

Proof. $M \geq_I 0 \Leftrightarrow JM \geq 0$ [By Theorem 3.2]

$$\Leftrightarrow \begin{pmatrix} J_m & 0 \\ 0 & J_n \end{pmatrix}\begin{pmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{pmatrix} \geq 0$$

$$\Leftrightarrow \begin{pmatrix} J_mA & J_nAX \\ J_mX^{[*]}A & J_nX^{[*]}AX \end{pmatrix} \geq 0$$

$$\Leftrightarrow \ J_mA \geq 0, J_nX^{[*]}A = (J_mAX)^*, (J_mA)(J_mA)^\dagger(J_mAX) = J_mAX$$

and $X^{[*]}AX - X^{[*]}AA^\dagger AX \geq 0$ [By Lemma 4.1]

$$\Leftrightarrow J_mA \geq 0, J_nX^{[*]}A = (AX)^*J_m\ ,\ J_mAA^\dagger AX = J_mAX \text{ and } X^{[*]}AA^\dagger AX \geq X^{[*]}AX.$$

$M \geq_I 0 \Leftrightarrow J_mA \geq 0,\ \ X^{[*]}A = (AX)^{[*]},\ AA^\dagger AX = AX$ and $X^{[*]}AA^\dagger AX \geq X^{[*]}AX$.

By using Theorem 3.2, Theorem 2.11 and Lemma 2.12, the above conditions reduces to

$M \geq_I 0 \Leftrightarrow J_mA \geq_I 0, X^{[*]}A = (AX)^{[*]}, Nu(A^{[*]}) \subseteq Nu((AX)^{[*]}$ and $X^{[*]}AA^\dagger AX \geq X^{[*]}AX$. ∎

Lemma 4.321 *Let* $M = \begin{pmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{pmatrix}$ *with* A *is* $J - EP$ *and* $X^{[*]}A = (AX)^{[*]}$ *then* $Nu(A^{[*]}) \subseteq Nu((AX)^{[*]}) \Leftrightarrow Nu(A) \subseteq Nu(X^{[*]}A)$

Proof. Since A is $J - EP$ then $Nu(A^{[*]}) \subseteq Nu((AX)^{[*]}) \Leftrightarrow Nu(A) = Nu(A^{[*]}) \subseteq Nu((AX)^{[*]}) = Nu\big(X^{[*]}A\big)$.

$Nu(A^{[*]}) \subseteq Nu((AX)^{[*]}) \Leftrightarrow Nu(A) \subseteq Nu(X^{[*]}A)$. ∎

Theorem 4.2 can be reformulated by using Lemma 4.3 as follows

Theorem 4.4 *Let* $M = \begin{pmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{pmatrix} \in \mathbb{C}^{(m+n)\times(m+n)}$ *then* $M \geq_I 0 \Leftrightarrow A \geq 0, X^{[*]}A = (AX)^{[*]}, Nu(A) \subseteq Nu(X^{[*]}A)$ *and* $X^{[*]}AA^\dagger AX \geq X^{[*]}AX$.

For $M = \begin{pmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{pmatrix}$ be an $(m+n)\times(m+n)$ positive semidefinite matrix with $X^{[*]}A = (AX)^{[*]}$, the Schur complement of A in M is $X^{[*]}AX - X^{[*]}AA^\dagger AX = S$ then the generalized inverse of M is given by $M^{[\alpha]} = \begin{pmatrix} A^{(\alpha)} + A^{(\alpha)}BS^{(\alpha)}CA^{(\alpha)} & A^{(\alpha)}BS^{(\alpha)} \\ -S^{(\alpha)}CA^{(\alpha)} & S^{(\alpha)} \end{pmatrix}$...(4.1), where $\alpha \in \{1,2,3,4\}$.

Theorem 4.522 *Let* $M = \begin{pmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{pmatrix}$ *be an* $(m+n)\times(m+n)$ *positive semidefinite matrix, Let* $M^{[\alpha]}$ *be the form (4.1), for* $\alpha = 1$, *if* $A^{(1)}$ *and* $S^{(1)}$ *are* $\{1\}$*-inverse of* A *and* S *respectively then* $M^{[1]}$ *is a* $\{1\}$*-inverse of* M *in* $\mathscr{I}$.

Proof. By Theorem 4.2, since $M \geq_I 0 \Leftrightarrow A \geq_I 0, X^{[*]}A = (AX)^{[*]}, N(A^{[*]}) \subseteq N((AX)^{[*]})$ and $X^{[*]}AA^\dagger AX \geq X^{[*]}AX$. Since $A \geq_I 0$, A is $J - EP$ from Definition 2.8 and Definition 3.1 it follows that $N(A) = N(A^{[*]})$. By using Lemma 4.3, we have $M \geq_I 0 \Leftrightarrow A \geq_I 0, N(A) \subseteq N(C), N(A^{[*]}) \subseteq N((AX)^{[*]})$ and $X^{[*]}AA^\dagger AX \geq X^{[*]}AX$. $A^\dagger$ is also a $\{1\}$-inverse of A.

For $\alpha = 1$ in (4.1).

We claim : $M^{[1]}$ is a $\{1\}$-inverse of M.

Using Theorem 2.11 and Lemma 2.12 for the above conditions we get $C = CA^\dagger A$ and $B = AA^\dagger B$.

$$\begin{aligned} M \circ M^{[1]} \circ M &= \begin{pmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{pmatrix}\begin{pmatrix} J_m & 0 \\ 0 & J_n \end{pmatrix}\begin{pmatrix} J_mA^{(1)}J_m + J_mXS^{(1)}X^{[*]}J_m & -J_mA^{(1)}AXS^{(1)}J_n \\ -J_nS^{(1)}X^{[*]}J_m & J_nS^{(1)}J_n \end{pmatrix} \\ &\quad \begin{pmatrix} J_m & 0 \\ 0 & J_n \end{pmatrix}\begin{pmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{pmatrix} \\ &= \begin{pmatrix} AJ_m & AXJ_n \\ X^{[*]}AJ_m & X^{[*]}AXJ_n \end{pmatrix}\begin{pmatrix} J_mA^{(1)}J_m + J_mXS^{(1)}X^{[*]}J_m & -J_mA^{(1)}AXS^{(1)}J_n \\ -J_nS^{(1)}X^{[*]}J_m & J_nS^{(1)}J_n \end{pmatrix} \\ &\quad \begin{pmatrix} J_mA & J_mAX \\ J_nX^{[*]}A & J_nX^{[*]}AX \end{pmatrix} \\ &= \begin{pmatrix} AA^{(1)}A & AA^{(1)}AX \\ X^{[*]}AA^{(1)}A & X^{[*]}AA^{(1)}AX \end{pmatrix} \\ &= \begin{pmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{pmatrix} \\ &= M \end{aligned}$$

Theorem 4.623 *Let* $M=\begin{pmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{pmatrix}$ *be an* $(m+n)\times(m+n)$ *positive semidefinite matrix, Let* $M^{[\alpha]}$ *be the form (4.1), for* $\alpha=(1,2)$, *if* $A^{(1,2)}$ *and* $S^{(1,2)}$ *are* $\{1,2\}$*-inverse of* A *and* S *respectively then* $M^{[1,2])}$ *is a* $\{1,2\}$*-inverse of* M *in* $\mathscr{J}$.

Proof. One can easily verify $M^{[1,2]}$ is a $\{1,2\}$-inverse of M. ■

Theorem 4.724 *Let* $M=\begin{pmatrix} A & AX \\ X^{[*]}A & X^{[*]}AX \end{pmatrix}$ *be an* $(m+n)\times(m+n)$ *positive semidefinite matrix, Let* $M^{[\alpha]}$ *be the form (4.1), for* $\alpha=(1,3)$, *if* $A^{(1,3)}$ *and* $S^{(1,3)}$ *are* $\{1,3\}$*-inverse of* A *and* S *respectively with* $S^{(1,3)}$ *Hermitian and* $rk(M)=rk(A)+rk(S)$. *Then* $M^{[1,3]}$ *is a* $\{1,3\}$*-inverse of* M *in* $\mathscr{J}$.

Proof. By Theorem 4.2, since $M\geq_I 0 \Leftrightarrow A\geq_I 0, X^{[*]}A=(AX)^{[*]}, N(A^{[*]})\subseteq N((AX)^{[*]})$ and $X^{[*]}AA^{\dagger}AX\geq X^{[*]}AX$. Since $A\geq_I 0$, A is $J-EP$ from Definition 2.8 and Definition 3.1 it follows that $N(A)=N(A^{[*]})$. By using Lemma 4.3, we have $M\geq_I 0\Leftrightarrow A\geq_I 0, N(A)\subseteq N(X^{[*]}A), N(A^{[*]})\subseteq N((AX)^{[*]})$ and $X^{[*]}AA^{\dagger}AX\geq X^{[*]}AX$. $A^{\dagger}$ is also a $\{1,3\}$-inverse of A. For $\alpha=1$ in (4.1).

By using Theorem 2.11 and Lemma 2.12 for the above conditions we get $X^{[*]}A=X^{[*]}AA^{\dagger}A$, $AX=AA^{\dagger}AX$ and $X^{[*]}AA^{\dagger}AX\geq X^{[*]}AX$.

$$M\circ M^{[1,3]}=\begin{pmatrix} AA^{(1,3)}J_m & 0 \\ X^{[*]}AA^{(1,3)}J_m-SS^{(1,3)}X^{[*]}AA^{(1,3)}J_m & SS^{(1,3)}J_n \end{pmatrix}$$

$$(M\circ M^{[1,3]})^{[*]}=\begin{pmatrix} J_m & 0 \\ 0 & J_n \end{pmatrix}\begin{pmatrix} AA^{(1,3)}J_m & 0 \\ X^{[*]}AA^{(1,3)}J_m-SS^{(1,3)}X^{[*]}AA^{(1,3)}J_m & SS^{(1,3)}J_n \end{pmatrix}^{*}\begin{pmatrix} J_m & 0 \\ 0 & J_n \end{pmatrix}$$

$$=\begin{pmatrix} J_m(AA^{(1,3)})^{[*]} & (X^{[*]}AA^{(1,3)}J_m-SS^{(1,3)}X^{[*]}AA^{(1,3)}J_m)^{[*]} \\ 0 & J_n(SS^{(1,3)})^{[*]} \end{pmatrix}.$$

$$rk(M\circ M^{[1,3]})^{[*]}=rk(\ J_m(AA^{(1,3)})^{[*]})+rk(J_m(SS^{(1,3)})^{[*]})=rk(AA^{(1,3)}J_m)+rk((SS^{(1,3)}J_n)$$
$$=rk(M\circ M^{[1,3]})=rk(A)+rk(S)=rk(M).$$ ■

Theorem 4.825 *Let* M *be an* $(m+n)\times(m+n)$ *positive semidefinite matrix, Let* $M^{[\alpha]}$ *be the form (4.1), for* $\alpha=(1,4)$, *if* $A^{(1,4)}$ *and* $S^{(1,4)}$ *are* $\{1,4\}$*-inverse of* A *and* S *respectively with* $S^{(1,4)}$ *Hermitian and* $rk(M)=rk(A)+rk(S)$. *Then* $M^{[1,4]}$ *is a* $\{1,4\}$*-inverse of* M *in* $\mathscr{J}$.

Proof. By Theorem 4.2, since $M\geq_I 0\Leftrightarrow A\geq_I 0, X^{[*]}A=(AX)^{[*]}, N(A^{[*]})\subseteq N((AX)^{[*]})$ and $X^{[*]}AA^{\dagger}AX\geq X^{[*]}AX$. Since $A\geq_I 0$, A is $J-EP$ from Definition 2.8 and Definition 3.1 it follows that $N(A)=N(A^{[*]})$. By using Lemma 4.3, we have $M\geq_I 0\Leftrightarrow A\geq_I 0, N(A)\subseteq N(X^{[*]}A), N(A^{[*]})\subseteq N((AX)^{[*]})$ and $X^{[*]}AA^{\dagger}AX\geq X^{[*]}AX$. $A^{\dagger}$ is also a $\{1,4\}$-inverse of A. For $\alpha=1$ in (4.1).

By using Theorem 2.11 and Lemma 2.12 for the above conditions we get $X^{[*]}A=X^{[*]}AA^{\dagger}A$, $AX=AA^{\dagger}AX$ and $X^{[*]}AA^{\dagger}AX\geq X^{[*]}AX$.

$$M^{[1,4]}\circ M=\begin{pmatrix} J_mA^{(1,4)}A & 0 \\ J_mA^{(1,4)}AX-J_mA^{(1,4)}AXS^{(1,4)}S & J_nS^{(1,4)}S \end{pmatrix}$$

$$(M^{[1,4]}\circ M)^{[*]}=\begin{pmatrix} J_m & 0 \\ 0 & J_n \end{pmatrix}\begin{pmatrix} J_mA^{(1,4)}A & 0 \\ J_mA^{(1,4)}AX-J_mA^{(1,4)}AXS^{(1,4)}S & J_nS^{(1,4)}S \end{pmatrix}^{*}\begin{pmatrix} J_m & 0 \\ 0 & J_n \end{pmatrix}$$

$$=\begin{pmatrix} (A^{(1,4)}A)^{[*]}J_m & (J_mA^{(1,4)}AX-J_mA^{(1,4)}AXS^{(1,4)}S)^{[*]} \\ 0 & (S^{(1,4)}S)^{[*]}J_n \end{pmatrix}.$$

$$rk(M^{[1,4]}\circ M)^{[*]}=rk((A^{(1,4)}A)^{[*]}J_m)+rk((S^{(1,4)}S)^{[*]}J_n)=rk(J_mA^{(1,4)}A)+rk(J_nS^{(1,4)}S)$$
$$=rk(M^{[1,4]}\circ M)=rk(A)+rk(S)=rk(M).$$ ■

ACKNOWLEDGEMENTS : The second author would like to express his sincere thanks to University Grants Commission for their financial support through RGNF scheme through grant no. F1-17.1/2016-17/RGNF-2015-17-SC-TAM-17791

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