On Detour Domination Number of a Graph

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Abstract
Let $G = (V, E)$ be a graph. A subset D of V is called a detour set of G if every vertex in $V - D$ lies in a detour joining a pair of vertices of D. A dominating set of G is a subset D of $V(G)$ such that every vertex in $V - D$ is adjacent to some vertex in D. In this paper, we study the detour domination number of some special graphs. We also introduce the new concept efficiently dominationating detour number, characterize it and derive the same of some standard and special graphs.

Keywords: Detour, Detour domination, Detour domination number.
AMS Subject Classification: 05C78.

1. Introduction

We consider finite graphs without loops and multiple edges. For any graph G, the set of vertices is denoted by $V(G)$ and the edge set by $E(G)$. The order and size of G are denoted by p and q respectively. We consider connected graphs with atleast two vertices. For basic definitions and terminologies, we refer to [3]. For vertices u and v in a connected graph G, the detour distance $D(u, v)$ is the length of the longest $u - v$ path in G. A $u - v$ path of length $D(u, v)$ is called a $u - v$ detour. It is known that the detour distance is a metric on the vertex set $V(G)$. A vertex x is said to lie on a $u - v$ detour P if x is a vertex of a $u - v$ detour path P including the verticles u and v. A set $S \subseteq V$ is called a detour set if every vertex v in G lies on a detour joining a pair of vertices of S. The detour number $dn(G)$ is called a minimum order of a detour set and any detour set of order $dn(G)$ is called a minimum detour set of G. These concepts were studied by Chartrand [4]. Let $G = (V, E)$ be a connected graph with at least two vertices. A set $S \subseteq V(G)$ is called a dominating set of G if every vertex in $V(G) - S$ is adjacent to some vertex in S. The domination number $\gamma(G)$ of G is the minimum order of its dominating sets and any dominating set of order $\gamma(G)$ is called γ-set of G. A detour domianating set is a subset S of $V(G)$ which is both a dominating and a detour set of G. A detour dominating set is said to be minimal detour dominating set of G if no proper subset of S is a detour dominating set of G. A detour dominating set S is said to be minimum detour dominating set of G if there exists no detour dominating set S' such that $|S'| < |S|$. The smallest cardinality of a detour dominating set of G is called the detour domination number of G. It is denoted by $\gamma_D(G)$. Any detour dominating set S of G of cardinality $\gamma_D(G)$ is called a (γ, D)-set of G.

Definition 1.1. Let $G = (V, E)$ be any graph and $v \in V(G)$. The neighbourhood of v, written as $N_G(v)$ or $N(v)$ is defined by $N(v) = \{x \in V(G): x \textit{ is adjacent to } v\}$.A vertex v in G is an extreme vertex of G if the subgraph induced by its neighbours is complete.

Definition1.2. [7] For each vertex v of a graph G, take a new vertex v' and join v' to all vertices of G adjacent to v. The graph $S(G)$ thus obtained is called the splitting graph of G.

Theorem 1.3. For a non-trivial tree, $dn(G) = k$, where k is the number of end vertices of G.

Theorem 1.4. The domination numbers of some standard graphs are given as follows.

(i) $\gamma(P_p) = \left\lceil\frac{p}{3}\right\rceil, p \geq 2.$

(ii) $\gamma(C_p) = \left\lceil\frac{p}{3}\right\rceil, p \geq 3$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x.

(iii) $\gamma(K_p) = \gamma(W_p) = \gamma(k_{1,n}) = 1.$

(iv) $\gamma(K_{m,n}) = 2 \; if \; m, n \geq 2.$

2. Detour domination number of some special graphs

Example2.1. Considering the graph G as in figure 2.1,

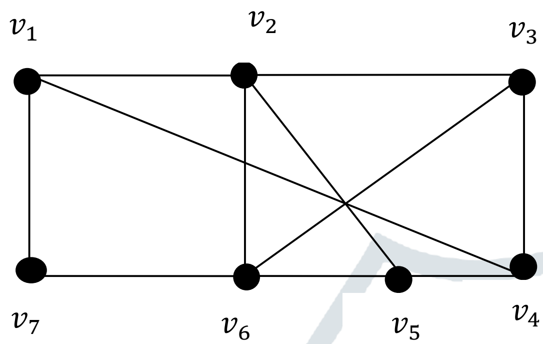

Figure 2.1

$\{v_1, v_2\}$ is a unique detour dominating set of G. So, $\gamma_D(G)$ = 2.

$\{v_1, v_2\}$is the dn-set of G. So $dn(G) = 2$. Also $\gamma(G) = 2$, G has more than one γ- set.

In this example, $\gamma_D(G) = dn(G) = \gamma(G)$.

Remark2.2. Let G be a connected graph with $p(\geq 2)$ vertices. Then, $\gamma(G) \leq \gamma_D(G)$. Strict inequality is also true in the above relation Considering P_{12}, $dn(P_{12}) = 2 \; and \; \gamma_D(P_{12}) = 6$. Therefore, $dn(P_{12}) < \gamma_D(P_{12})$.

Remark2.3. In general, $\gamma_D(G)$, $dn(G)$ and $\gamma(G)$, all need not be equal. For example, consider P_9, $\gamma_D(P_9) = 4$, $dn(P_9) = 2$ and $\gamma(P_9) = 3$. Further, $\gamma_D(P_7) = \gamma(P_7) = 3$. But, $dn(P_7) = 2$. Also, $\gamma_D(P_3) = dn(P_3) = 2$. whereas, $\gamma(P_3) = 1$.

Observations 2.4. Let $G = (V, E)$ be any connected graph with at least two vertices. Then,

1. $\gamma_D(G) \geq dn(G)$ and $\gamma_D(G) \geq \gamma(G)$.
2. If G is a graph with at least one pendent vertex, then for every (γ, D)-set D of $G, V - D$ is not a (γ, D)-set of G.
3. Every super set of a (γ, D)-set of G is a (γ, D)-set of G.
4. $\gamma_D(K_n) = 2$Further, $\gamma(K_n) < dn(K_n) = \gamma_D(K_n) = 2$.

Theorem 2.5. $\gamma_G[S(K_n)] = n$ where $S(K_n)$ denotes the splitting graph of K_n.

Proof. Suppose V and V' are the vertex sets of K_n and $S(K_n)$ respectively. Let S be a detour dominating set of $S(K_n)$.As $V' - V$ is independent and K_n is complete, every vertex v of $V' - V$ lie in S, otherwise v' does not lie on any detour joining the vertices of S. Therefore, S contains all the vertices of $V' - V$. That is, $V' - V \subseteq S$. Further, the vertices of $V' - V$dominate and detourdominate all the vertices of $S(K_n)$. Therefore, $V' - V$ is a unique minimum (γ, D)-set of $S(K_n)$ and so $\gamma_G[S(K_n)] = |V' - V| = |V'|/2 = n$.

Example 2.6. Consider the splitting graph of K_4 in figure2.1, $\{u_1, u_2, u_3, u_4\}$ is the unique minimum detour dominating set of $S(K_4)$ and so $\gamma_D[S(K_4)] = 4$.

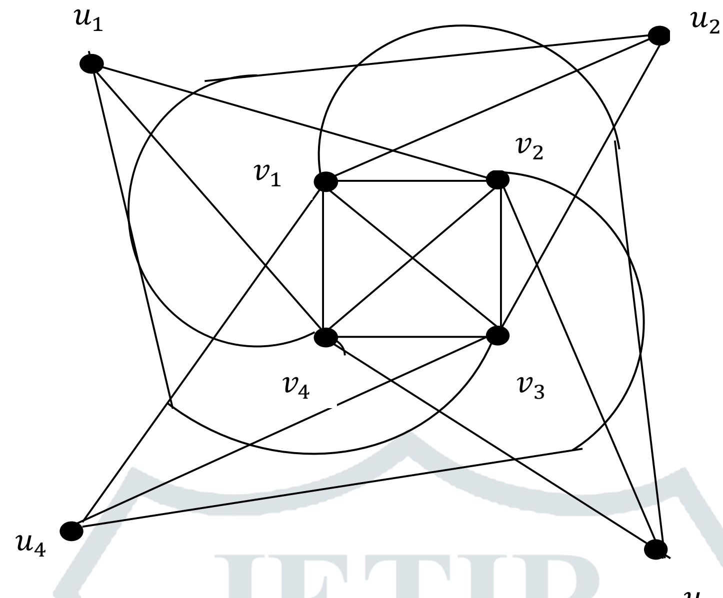

Figure 2.2

Theorem 2.7. Let $F_n$be the fan graph obtained from a path P_n by adding a new vertex and joining it to all the vertices of the path by an edge.

Then, $\gamma_D(F_n) = \begin{cases} 3 & if\ n \geq 5 \\ 2 & if\ n\ 2,3, or\ 4 \end{cases}$

Proof: Let $V(F_n) = \{v, v_1, v_2, \dots, v_n\}$. $S = \{v_1, v, v_n\}$ is a minimum detour dominating set of F_n. It is observed that $d(v, v_i) = 1$ for all i and the vertex v lies in a detour joining every pair of non-adjacent vertices of V. Also it is dominating by every vertex of V.

Case($\boldsymbol{i}$): If $n \geq 5$

Define $V(F_5) = \{v, v_1, v_2, \dots, v_5\}$ and $S = \{v_1, v, v_5\}$. Clearly, S is the unique minimum detour dominating set of F_n and $|S| = 3$.

Case($\boldsymbol{ii}$): If $n = 2,3, or\ 4$.

Let $S = \{v_1, v_n\}$ be the unique minimum detour dominating set of F_n and $|S| = 2$.

Therefore, $\gamma_D(F_n) = \begin{cases} 3 & if\ n \geq 5 \\ 2 & if\ n = 2,3, or\ 4 \end{cases}$

Example 2.8. Consider the fan graph F_9 in figure2.2, $\{v_1, v, v_9\}$ is the unique minimum detour dominating set of $F_9$and so $\gamma_D(F_9) = 3$.

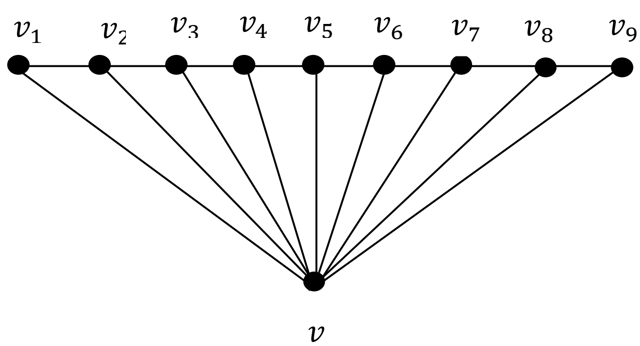

Figure 2.2

Theorem 2.9. Let $(C_n \odot K_1)$ be the crown obtained by joining a pendent edge to each vertex of C_n. Then, for $n \geq 3, \gamma_D(C_n \odot K_1) = n$.

Proof: Let $\{u_1, u_2, \ldots, u_n\}$ be set of end vertices of the crown obtained by joining a pendent edge to each vertex of $C_n \odot K_1$. If S is a detour dominating set of $C_n \odot K_1$, then by remark2.5, $S' \subseteq S$. Further, all the elements in $C_n \odot K_1$ are detour dominated by the elements of S'. Thus, by theorem 1.3, S′ is the unique minimum (γ, D)- set of $C_n \odot K_1$. Therefore, $\gamma_D(C_n \odot K_1) = |S'| = n$.

Example 2.10. Consider the crown $C_5 \odot K_1$ in figure (2.3), $\{u_1, u_2, u_3, u_4, u_5\}$ is the unique minimum (γ, D) −set of $C_5 \odot K_1$ and so $\gamma_D(C_5 \odot K_1) = 5$.

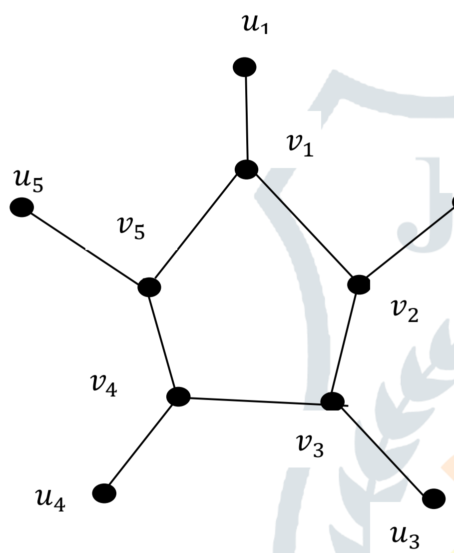

Figure 2.3

Theorem 2.11. Let AC_n be the armed crown in which path P_2 is attached at each vertex of cycle C_n by an edge, where n is the number of vertices in cycle C_n. Then, for $n \geq 3, \gamma_D(AC_n) = n + \left\lceil \frac{n}{3} \right\rceil$.

Proof: Let V and V' be the vertex sets of C_n and AC_n respectively. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $V' = \{u_i, v_i, w_i : 1 \leq i \leq n\}$ such that each u_i is adjacent to both v_i and pendant vertex w_i for all i. Let $S' = \{w_1, w_2, \ldots, w_n\}$. Then, any detour dominating set S of AC_n contains S'. Clearly, S' is the unique detour dominating set of AC_n. Now, any dominating set of C_n together with S′ forms a minimum detour dominating set of AC_n. Therefore, $\gamma_D(AC_n) = n + \gamma(C_n) = n + \left\lceil \frac{n}{3} \right\rceil$.

Example 2.12. Consider the armed crown AC_6 in figure 2.4, $w_1, w_2, w_3, w_4, w_5, w_6, v_1, v_4\}$, $\{w_1, w_2, w_3, w_4, w_5, w_6, v_2, v_5\}$ and $\{w_1, w_2, w_3, w_4, w_5, w_6, v_3, v_6\}$ are the minimum detour dominating sets of AC_6 and so, $\gamma_D(AC_6) = 6 + \left\lceil \frac{6}{3} \right\rceil = 8$.

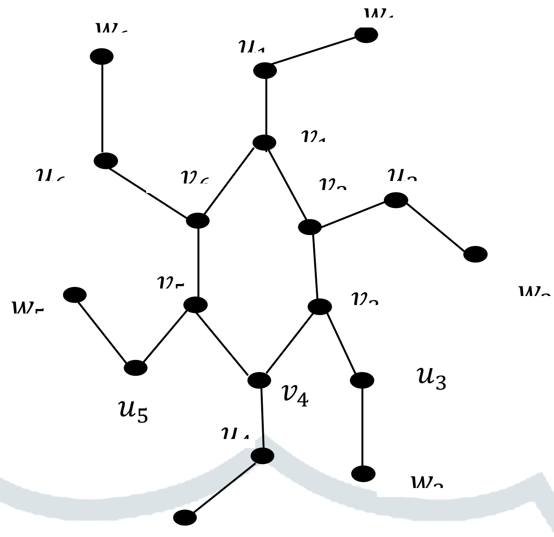

Figure 2.4

Theorem 2.13. Let H_n be the helm graph obtained from a wheel W_n by attaching a pendant edge to each rim vertex. Then, for $n \geq 4$, $\gamma_D(H_n) = n$.

Proof: Let V and V' be the vertex sets of W_n and H_n respectively. Let $V = \{ v_1, v_2, \dots, v_{n-1}, v\}$with v as central vertex and $V' = V \cup \{u_1, u_2, \dots, u_{n-1}\}$ such that each u_i is adjacent to v_i $(1 \leq i \leq n-1)$. Let S be a detour dominating set of H_n. Then, by remark2.5, $\{u_1, u_2, \dots, u_{n-1}\} \subseteq S$. Take $\{u_1, u_2, \dots, u_{n-1}\} = X$. Clearly, for $n \geq 4$, every vertex of $V' - X$ lies in a detour joining the vertices of X. But, the central vertex v is not dominated by X. Thus $X \cup \{v\}$ and $X \cup \{v_i\}$, $1 \leq i \leq n-1$ are minimum detour dominating sets of H_n.

Therefore,$\gamma_D(H_n) = |X| + 1 = n - 1 + 1 = n.$

Example 2.14. Consider the helmH_6 in figure2.5, $\{ u_1, u_2, u_3, u_4, u_5, v_1\}$ is a minimum detour dominating set of H_6 and so $\gamma_D(H_6) = 6$.

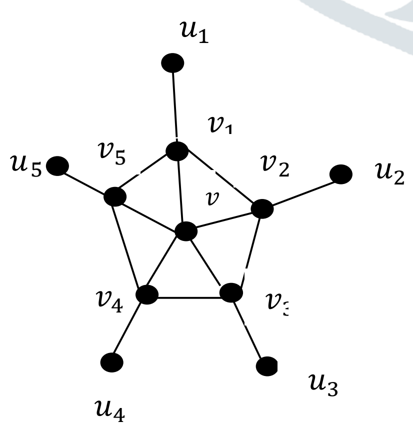

Figure 2.5

Theorem 2.15. For a flower graph $Fl_n, \gamma_D(Fl_n) = n$.

Proof. The set of vertices of inner cycle C_n forms a unique minimum detour dominating set of Fl_n.

Therefore, $\gamma_D(Fl_n) = n$.

Example 2.16. Consider the flower Fl_6 in figure 2.6, $\{v_1, v_2, v_3, v_4, v_5\}$ is a minimum (γ, D)--set of Fl_6 and so $\gamma_D(Fl_6) = 6$.

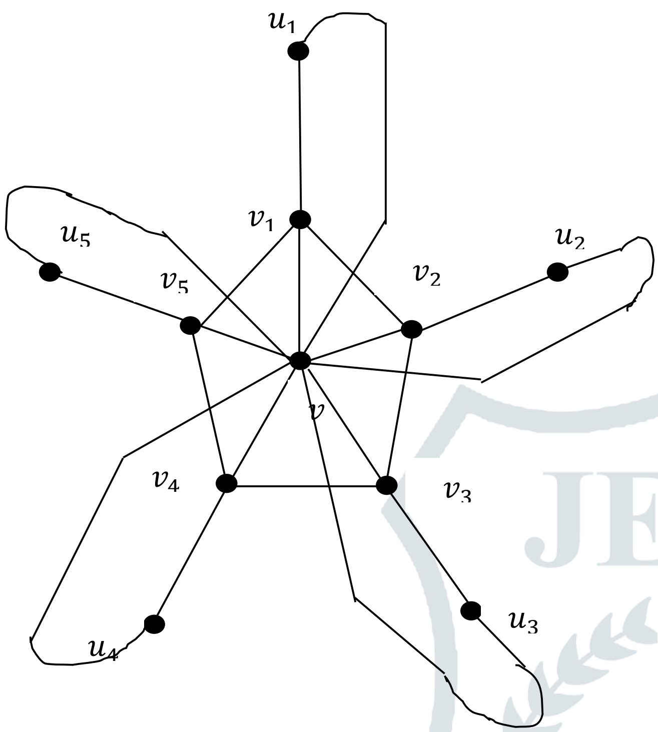

Figure 2.6

Theorem 2.17. Let Wb_n be web graph obtained by joining the pendant vertices of a helm H_n to form a cycle and then adding a pendant edge to each vertex of outer cycle. Then, for $n \geq 4$, $\gamma_D(Wb_n) = n$.

Proof: Let $V(Wb_n) = \{v, v_i, u_i, w_i : 1 \leq i \leq n-1\}$ with v as its central vertex and $E(Wb_n) = \{v\, v_i, v_i u_i, u_i w_i : 1 \leq i \leq n-1\} \cup \{v_{n-1}\, v_1, u_{n-1} u_1\} \cup \{v_i\, v_{i+1}, u_i u_{i+1} : 1 \leq i \leq n-2\}$. Let S be a (γ, D)set of Wb_n. Then, by remark 2.5, $\{w_1, w_2, \dots, w_{n-1}\} \subseteq S$. Define $X = \{w_1, w_2, \dots, w_{n-1}\}$ and $S = X \cup \{v\}$. Clearly, S is the unique minimum (γ, D)--set of Wb_n and so$|S| = |X \cup \{v\}| = n - 1 + 1$. Hence, , $\gamma_D(Wb_n) = n$.

Example 2.18. Consider the web Wb_8 in figure (2.7), $\{w_1, w_2, w_3, w_4, w_5, w_6, w_7, v\}$ is the unique minimum detour dominating set of Wb_8 and so $\gamma_D(Wb_8) = 8$.

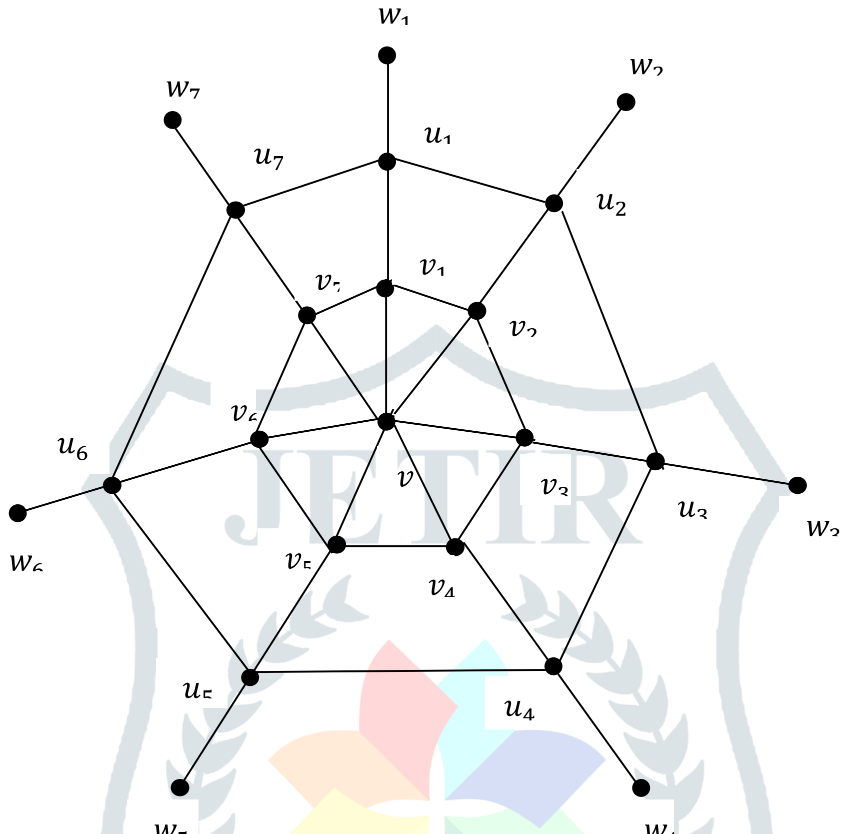

Figure 2.7

3. Efficiently Dominating Detour Domination Number of Graphs

Definition 3.1. Let G be a connected graph. An efficiently dominating (γ, D)-set of G is a detour dominating set of G such that for every $v \in V(G)$, $|N[v] \cap S| = 1$. The minimum cardinality among all efficient dominating detour dominating sets is called the efficiently dominating (γ, D)- number of G and is denoted by $e(\gamma, D)$. An efficiently dominating (γ, D)-set of cardinality $e\gamma_D(G)$ is called a $e\gamma_D$-set of G. A graph G is said to be an efficiently dominating (γ, D)-graph if it has an efficiently dominating (γ, D)-set.

Example 3.2. For the graph G in figure 3.1(a), $S = \{v_1, v_5\}$ is one of the minimum detour dominating set of G. Also, $|N[v_i] \cap S| = 1$ for all $v_i \in V(G)$, $1 \leq i \leq 6$. Theorefore, S is the minimum efficiently dominating (γ, D)-set of G and so $e\gamma_D(G) = 2$. Therefore, the graph G is an efficiently dominating (γ, D)-graph.

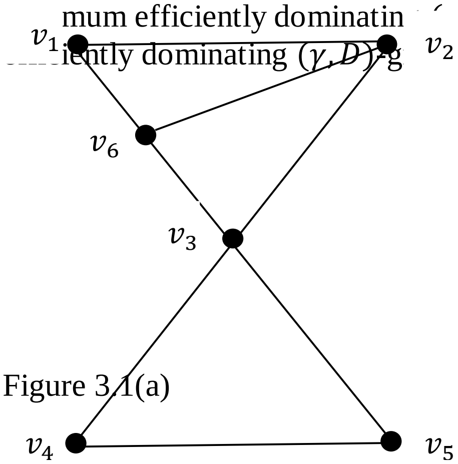

Figure 3.1(a)

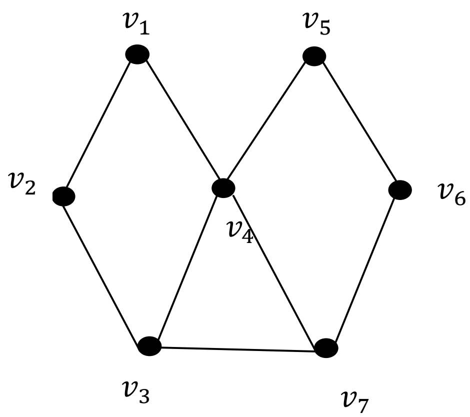

Figure 3.1(b)

For the graph G in figure 3.1(b), $S = \{v_1, v_5, v_7\}$ is a detour dominating set of G. But not an efficiently dominating (γ, D)-set. Therefore, the graph G is only a detour dominating graph.

Theorem 3.3. Every end vertex of G belongs to every efficiently dominating (γ, D)-graph.

Proof. Every efficiently dominating detour dominating set of G is a detour dominating set of G. Therefore, by theorem 1.1 and 1.2, every end vertex of G belongs to every efficiently dominating (γ, D)-graph.

Observation 3.4.(1) If G is a graph with a vertex v which is a support vertex of at least two end vertices, then G does not have an efficiently dominating detour domination set.

(2) Every detour dominating set need not be an efficiently dominating detour dominating set.

Theorem 3.5. For a graph G of order $p > 2, \gamma(G) \leq \gamma_D(G) \leq e\gamma_D(G)$.

Proof. Since every detour dominating set of G is also a dominating set and every efficiently dominating detour dominating set is also a detour dominating set, $\gamma(G) \leq \gamma_D(G) \leq e\gamma_D(G)$.

Theorem 3.6. Every efficient dominating detour domination set is independent.

Proof. Let G be a graph and S be a efficiently dominating detour domination set. Let $u, v \in S$. Suppose u and v are adjacent. Then $|N[v] \cap S| \neq 1$ and $|N[u] \cap S| \neq 1$. So, S is not an efficiently dominating detour domination set of G, which is a contradiction. Therefore, u and v are not adjacent. Hence, S is independent. Hence, every efficiently dominating detour domination set is independent.

Theorem 3.7. If S is an efficiently dominating detour domination set of a connected graph G then $G - S$ is a dominating set of G.

Proof. Since every efficiently dominating detour domination set is independent and G is connected, every vertex in S is adjacent to at least one vertex in $V - S$. Therefore, $V - S$ is dominating set of G.

Observation 3.8. If S is an efficiently dominating detour domination set of a graph G with end vertices, then $V - S$ is not a detour dominating set of G.

Theorem 3.9. P_n is an efficiently dominating detour dominating graph iff $n = 3k + 1,\ k = 1,2, \ldots$ and $e\gamma_D(P_n) = k + 1\ if\ n = 3k + 1$.

Proof. Let $G \cong P_{3n+1},\ n \in N$.

Case1: $n = 3k + 1$.Let $V(G) = \{v_1, v_2, \ldots, v_{3n+1}\}$ $S = \{v_1, v_4, \ldots, v_{3n+1}\}$ is the unique minimum detour dominating set of P_{3n+1}. Further $|N[v] \cap S| = 1$ for all $v \in 3n + 1$. Therefore, S is the unique efficiently dominating detour dominating set of P_{3n+1}. Therefore, $e\gamma_D(P_{3n+1}) = \gamma_D(P_{3n+1})$. Hence $e\gamma_D(P_n) = k + 1$.

Case2: $n = 3k$, or $n = 3k + 2$, In each case, every detour dominating set of P_n contains at least two vertices u and v such that $d(u, v) \leq 2$. Therefore, u or v or any intermediate vertex in a $u - v$ shortest path fails to satisfy the efficient domination condition. Hence, P_n has no efficiently dominating detour domination set when $n = 3k$ or $3k + 2$.

Theorem 3.10. C_{3n} is a efficiently dominating detour dominating graph and $e\gamma_D(C_{3n}) = n\ if\ n > 1, n \in N$.

Proof. $n = 3k$. Let $V(C_{3n}) = \{v_1, v_2, \dots, v_{3n}\}$. The only detour dominating sets are $S_1 = \{v_1, v_4, \dots, v_{3(n-1)+1}\}$, $S_2 = \{v_2, v_5, \dots, v_{3(n-1)+2}\}$ and $S_3 = \{v_3, v_6, \dots, v_{3n}\}$. Further $|N[v] \cap S_i| = 1$ for all $v \in C_{3n}$and $i = 1,2,3$. Hence, C_{3n} is a efficiently dominating detour dominating graph andS_i, $i = 1,2,3$ are efficiently dominating detour dominating sets of C_{3n}. Therefore, $e\gamma_D(C_{3n}) = \gamma_D(C_{3n})$. Therefore, by Theorem 1.2,$e\gamma_D(C_{3n}) = \gamma_D(C_{3n}) = \left\lceil \frac{3n}{3} \right\rceil = n$.

Theorem 3.11. Complete graph $K_{n\,,}\ n > 2$, are not efficiently dominating detour dominating graphs.

Proof. By Observations 3.4 (2), any efficiently dominating detour domination set is also a detour domination set. Further, any detour domination set of K_n contains at least two vertices. As being vertices of K_n they are adjacent. Therefore, by observation 3.4(2) , K_p has no efficiently dominating detour dominating sets.

Corollary 3.12. Complete bipartite graphs are not efficiently dominating detour domination graphs.

Proof . Let V_1, V_2 be the bipartition of $V(K_{m,n})$. Any detour domination set of $K_{m,n}$ contains at least one vertex from both V_1 and V_2. Obviously they are adjacent. Therefore, by Theorem 3.6. complete bipartite graphs are not efficiently dominating detour domination graphs.

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