$\rho - T_0$ AND $\rho - T_1$ SPACES IN TOPOLOGICAL SPACES

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Abstract: A subset A of a space (X,τ) is said to be ρ-closed if $pcl(A) \subseteq Int(U)$ whenever $A \subseteq U$ and U is $\tilde{g} - open$ in (X,τ). The complement of ρ-closed set is ρ-open set. A function $f: (X, \tau) \to (Y, \sigma)$ is said to be ρ-continuous if $f^{-1}(V)$ is ρ-closed in (X,τ) for every closed set V in (Y,σ). A function $f: (X, \tau) \to (Y, \sigma)$ is said to be ρ-irresolute if $f^{-1}(V)$ is ρ-closed in (X,τ) for every ρ-closed set V in (Y,σ). In this paper, we introduce two new topological spaces namely $\rho - T_0$ and $\rho - T_1$ spaces via ρ-open sets. Also we characterize their properties.

Keywords: ρ-closed set, ρ-open set, $\rho - T_0$ space, $\rho - T_1$ space, ρ-D set, ρ-symmetric and ρ-kernel.

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1. Introduction:

The word "topology" is derived from two Greek words, topos meaning 'surface' and logos meaning 'discourse' or 'study'. Topology thus literally means the study of surfaces. The study of topological spaces, their continuous mappings and general properties make up one branch of topology known as "general topology". The fundamental ideas in general topology are those of convergence and continuity. The separation axioms that were studied together in this way were the axioms for Hausdorff spaces, regular spaces and normal spaces. Separation axioms and closed sets in topological spaces have been very useful in the study of certain objects in digital topology [4,5]. The concept of $T_i - spaces$ (i=0,1,2) was defined by Willard in 1970 [8]. In 2012, C.Devamanoharan [2] introduced and studied ρ-closed set in Topological Spaces. Now we introduce two new topological spaces namely $\rho - T_0$ and $\rho - T_1$ spaces.

2. Preliminaries:

Definition 2.1:[2] A subset A of a space (X,τ) is said to be ρ-closed if $pcl(A) \subseteq Int(U)$ whenever $A \subseteq U$ and U is $\tilde{g} - open$ in (X,τ).

Definition 2.2:[6] A topological space (X,τ) is said to be T_0 space if for each pair of distinct points x and y in X, there exists a open set U of X such that either $x \in U \; and \; y \notin U$ or $x \notin U \; and \; y \in U$.

Definition 2.3:[6] A topological space (X,τ) is said to be T_1 space if for each pair of distinct points x and y in X, there exists two open sets U and V such that $x \in U\ and\ y \notin U$ or $x \notin V\ and\ y \in V$.

Proposition 2.4: [2] Let A and B be subsets of a topological space (X,τ). Then

1. If $A \subseteq B$, then $\rho - cl(A) \subseteq \rho - cl(B)$
2. $\rho - cl\big(\rho - cl(A)\big) = \rho - cl(A)$.

Proposition 2.5: [2]

Let (X,τ) be a topological space. Then $x \in \rho - cl(A)$ if and only if $U \cap A \neq \varphi$ for every $\rho - open\ set\ U$ containing x.

3. $\rho - T_0$ AND $\rho - T_1$ SPACES

Definition 3.1: A topological space (X,τ) is said to be ρ-T_0 if for each pair of distinct points x,y in X, there exists a ρ-open set U of X such that either $x \in U\ and\ y \notin U$ or $x \notin U\ and\ y \in U$.

Definition 3.2: A topological space (X,τ) is said to be ρ-T_1 if for each pair of distinct points x,y in X, there exists two ρ-open sets U and V such that $x \in U\ and\ y \notin U$ or $x \notin V\ and\ y \in V$.

Definition 3.3: A subset A of a topological space X is called ρ-difference set (briefly ρ-D set) if there are $U, V \in \rho - open\ set\ of\ X$ such that $U \neq X\ and\ A = U \backslash V$. It is true that every ρ-open set U different from X is a ρ-D set if A=U and V=ϕ.

Definition 3.4: A topological space (X,τ) is said to be ρ-symmetric if for x,y in X, $x \in \rho - cl(\{y\})$ implies $y \in \rho - cl(\{x\})$ (or $x \notin \rho - cl(\{y\})$ implies $y \notin \rho - cl(\{x\})$.

Definition 3.5: Let (X,τ) be a topological space and $A \subseteq X$. Then the ρ-kernel of A is denoted by ρ-ker(A) and is defined to be $\rho - \ker(A) = \cap \{U \in \rho - open\ set\ of\ X : A \subseteq U\}$.

Definition 3.6: A topological space (X,τ) is called a ρ-Hausdroff if for each pair x, y of distinct points of X, there exists ρ-open neighbourhoods U_1 and U_2 of x,y respectively, that are disjoint.

Theorem 3.7: A topological space (X,τ) is ρ-T_0 if and only if for each pair of distinct points x,y of X, $\rho - cl(\{x\}) \neq \rho - cl(\{y\})$.

Proof: Let (X,τ) be a ρ-T_0 space and x,y be any two distinct points of X. There exists a ρ-open set U containing x or y, say x but not y. Then X\U is a ρ-closed set which does not contain x but contains y. Since ρ-cl({y}) is the smallest ρ-closed set containing y, ρ-cl({y})⊆X\U. Therefore, $x \notin \rho - cl(\{y\})$. Consequently, $\rho - cl(\{x\}) \neq \rho - cl(\{y\})$. Conversely, suppose that $x, y \in X, x \neq y\ and\ \rho - cl(\{x\}) \neq \rho - cl(\{y\})$. Let $z \in X$ such that $z \in \rho - cl(\{x\})\ but\ z \notin \rho - cl(\{y\})$. We claim that $x \notin \rho - cl(\{y\})$. Suppose $x \in \rho - cl(\{y\})$. Then by proposition 2.4, $\rho - cl(\{x\}) \subseteq \rho - cl(\rho - cl(\{y\}) = \rho - cl(\{y\})$. Therefore, $z \in \rho - cl(\{y\})$ which is a contradiction. Thus, $x \notin \rho - cl(\{y\})$. Now, $X \backslash \rho - cl(\{y\})$ is a ρ-open set in X such that $x \in X \backslash \rho - cl(\{y\})andy \notin X \backslash \rho - cl(\{y\})$. Hence (X,τ) is a ρ-$T_0$ space.

Theorem 3.8: A topological space (X,τ) is ρ-T_1 if and only if the singletons are ρ-closed sets.

Proof: Let (X,τ) be a ρ-T_1 space and x be any point of X. Suppose $y \in X\backslash\{x\}$. Then $x \neq y$. So, there exists a ρ-open set U such that $y \in U\ but\ x \notin U$. Consequently, $y \in U \subseteq X\backslash\{x\}$, that is $X\backslash\{x\} = \cup\{U: y \in X\backslash\{x\}\}$ which is ρ-open. So, singletons in (X,τ) are ρ-closed sets. Conversely, suppose {p} is ρ-closed for every $p \in X$. Let $x, y \in X\ with\ x \neq y$. Since $x \neq y, y \in X\backslash\{x\}$. So, X\{x} is a ρ-open set contains y but not x. Similarly, X\{y} is a ρ-open set contains x but not y. Hence, X is a ρ-T_1 space.

Theorem 3.9: Every ρ-T_1 space is ρ-T_0 space.

Proof: The proof follows from definitions 3.1 and 3.2.

Remark 3.10: The following example shows that the converse of the above theorem need not be true.

Example 3.11: Consider the space X={a,b,c} and τ ={X,ϕ,{a},{a,c}}. Clearly (X,τ) is a ρ-T_0 space but not ρ-T_1 space.

Theorem 3.12: Every ρ-T_2 space is ρ-T_1 space.

Proof: The proof follows from definition 3.2 and 3.6.

Theorem 3.13: Every proper ρ-open set is a ρ-D set.

Remark 3.14: The following example shows that the converse of the above theorem need not be true.

Example 3.15: Let X={a,b,c} with the topology τ ={X,ϕ,{c},{b,c}}. Consider U = {a,c} ≠ X and V = {b,c} are ρ-open sets in X. Then we have A = U\V = {a} is a ρ-D set but not a ρ-open set.

Theorem 3.16: Let (X,τ) be a topological space. Then (X,τ) is a ρ-symmetric space if and only if {x} is ρ-closed for each $x \in X$.

Proof: Assume that {x} is ρ-open in X. Then there exists ρ-open set U such that $\{x\} \subseteq U$, but $\rho - cl(\{x\}) \nsubseteq U$. Then $\rho - cl(\{x\}) \cap (X\backslash U) \neq \varphi$. Now, we take $y \in \rho - cl(\{x\}) \cap (X\backslash U)$, then by hypothesis $x \in \rho - cl(\{y\})$ and also $\rho - cl(\{y\}) \subseteq (X\backslash U)$. Therefore, $x \notin U$, which is a contradiction. Hence, {x} is ρ-closed for each $x \in X$. Conversely, suppose singleton sets are ρ-closed in X. We claim that X is a ρ-symmetric space. Assume that $x \in \rho - cl(\{y\})\ but\ y \notin \rho - cl(\{x\})$. Then $\{y\} \subseteq X\backslash\rho - cl(\{x\})$. So, $\rho - cl(\{y\}) \subseteq X\backslash\rho - cl(\{x\})$. Thus, $x \in X\backslash\rho - cl(\{x\})$. Therefore, $x \notin \rho - cl(\{x\})$ which is a contradiction. Hence, $y \in \rho - cl(\{x\})$.

Theorem 3.17: If a topological space (X,τ) is a ρ-T_1 space, then it is ρ-symmetric.

Proof: Since X is a ρ-T_1 space and by theorem 3.8, singleton sets are ρ-closed. Therefore by theorem 3.16, (X,τ) is a ρ-symmetric space.

Theorem 3.18: If a topological space (X,τ) is ρ-symmetric and ρ-T_0 space, then (X,τ) is ρ-T_1 space.

Proof: Let $x \neq y$ and (X,τ) be a ρ-T_0 space.

We may assume that $x \in U \subseteq X\backslash\{y\}$ for some $U \in \rho - open\ set\ of\ X$. Then $x \in \rho - cl(\{y\})$. Since X is a ρ-symmetric space, $y \notin \rho - cl(\{x\})$. Then there exists a ρ-open set V such that $y \in V \subseteq X\backslash\{x\}$. Hence, (X,τ) is a ρ-$T_1$ space.

Theorem 3.19: Let (X,τ) be a topological space and $x \in X$. Then $y \in \rho - \ker(\{x\})$ if and only if $x \in \rho - cl(\{y\})$.

Proof: Suppose that $y \notin \rho - ker(\{x\})$. Then there exists a ρ-open set V containing x such that $y \notin V$. By proposition 2.5, $x \notin \rho - cl(\{y\})$. Conversely, assume that $x \notin \rho - cl(\{y\})$. Then there exists a ρ-open set U containing x such that $y \notin U$. By the definition of ρ-kernel, $y \notin \rho - \ker(\{x\})$.

Theorem 3.20: Let (X,τ) be a topological space and A be a subset of (X,τ). Then $\rho - \ker(A) = x \in X{:}\, \rho - cl(\{x\}) \cap A \neq \varphi\}$.

Proof: Let $x \in \rho - \ker(A)$. Suppose $x \notin \{x \in X{:}\, \rho - cl(\{x\}) \cap A \neq \varphi\}$. Then $\rho - cl(\{x\}) \cap A = \varphi$. So, X\ρ-cl({x}) is a ρ-open set containing A and $x \notin X\backslash\rho - cl(\{x\})$. Therefore, by definition of ρ-kernel, $x \notin \rho - \ker(A)$. It contradicts $x \in \rho - \ker(A)$. Conversely, if $\rho - cl(\{x\}) \cap A \neq \varphi$.

Suppose that $x \notin \rho - \ker(A)$. Then there exists a ρ-open set V containing A such that $x \notin V$. Now, let $y \in \rho - cl(\{x\}) \cap A$. Then $y \in \rho - cl(\{x\})$ and $y \in A$. By proposition 2.5, $y \in \rho - cl(\{x\})$ implies $V \cap \{x\} \neq \varphi$ for every open set V containing y. Hence, $x \in V$. By this contradiction, $x \in \rho - \ker(A)$.

Theorem 3.21: The following properties hold for the subsets A, B of a topological space (X,τ):

1. $A \subseteq \rho - \ker(A)$.
2. $A \subseteq B$ implies $\rho - \ker(A) \subseteq \rho - \ker(B)$.
3. If A is ρ-open in (X,τ), then $A = \rho - \ker(A)$.
4. $\rho - \ker(\rho - \ker(A)) = \rho - \ker(A)$.

Proof:

1. Suppose that A is any subset of X. If $x \notin \rho - \ker(A)$, then there exists $U \in \rho - open\ set\ of\ X$ such that $A \subseteq U$ and $x \notin U$. Therefore, $x \notin A$.
   Hence, $A \subseteq \rho - \ker(A)$.
2. Let $A \subseteq B$. Suppose $\rho - \ker(A) \nsubseteq \rho - \ker(B)$. Then $x \in \rho - \ker(A)$ but $x \notin \rho - \ker(B)$. By the definition of ρ-kernel, there exists a ρ-open set U such that $B \subseteq U\ and\ x \notin U$. Since $A \subseteq B \subseteq U, x \notin \rho - \ker(A)$. By this contradiction, $\rho - \ker(A) \subseteq \rho - \ker(B)$.
3. Obvious from the definition of ρ-ker(A).
4. From (1) & (2), we have $\rho - \ker(A) \subseteq \rho - \ker(\rho - \ker(A))$. To prove the other implication, if $x \notin \rho - \ker(A)$, then there exists $U \in \rho - open\ set\ of\ X$ such that $A \subseteq U\ and\ x \notin U$. Therefore, $\rho - \ker(A) \subseteq U$ and so $x \notin \rho - \ker(\rho - \ker(A))$. Hence $\rho - \ker(A) = \rho - \ker(\rho - \ker(A))$.

Theorem 3.22: If a singleton set {x} is a ρ-D set of (X,τ), then $\rho - \ker(\{x\}) \neq X$.

Proof: Since {x} is a ρ-D set of (X,τ), there exists two ρ-open subsets U, V such that {x}=U\V. So, $\{x\} \subseteq U and\ U \neq X$. By theorem 3.21 (2) & (3), $\rho - \ker(\{x\}) \subseteq U \neq X$. Hence, $\rho - \ker(\{x\}) \neq X$.

References:

[1] Bhattacharya.P, A.Kar, "Some weak separation axioms", Bull.Cal.Math.Soc., 82(1990), 415-422.

[2] Devamanoharan.C, S.PiousMissier & S.Jafari, "$\rho - closed$ sets in Topological Spaces", European Journal of Pure and Applied Mathematics, vol.5, No.4, 2012, 554-566.

[3] Hariwan, Z.Ibrahim, "BC-Separation Axioms in Topological spaces", Gen.Math.Notes, vol17, july 2013, 45-62.

[4] Khalimsky,E.D, "Applications of Connected Ordered Topological Spaces in Topology", Conference of Mathematics Departments of Povolsia(1970).

[5] Kong.T.Y, R.Kopperman, P.R.Meyer, "A Topological Approach to Digital Toplogy", Amer.Math.Monthly, 98(1991), 901-917.

[6] Levine.N, "Semi open sets, semi-continuity in topological spaces", Amer.Math.Monthly, 70, 36- 41.1963.

[7] Sundaram.P and Sheik John.M, "On ω-closed sets in topology", Acta Ciencia Indica 4, 389-392.2000.

[8] Willard.S, "General Topology", Addisen Wesley, 106-107.1970.